

Structures of Malcev Bialgebras on a simple non-Lie Malcev algebra.

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Abstract

In this work, we consider Malcev bialgebras. We describe all structures of a Malcev bialgebra on a simple non-Lie Malcev algebra.

Key words: Lie bialgebra, Malcev bialgebra, classical Yang-Baxter equation, nonassociative coalgebra, simple non-Lie Malcev algebra

Lie bialgebras are Lie algebras and Lie coalgebras at the same time, such that comultiplication is a 1-cocycle. These bialgebras were introduced by Drinfeld [1] in studying the solutions to the classical Yang-Baxter equation. In [2, 3], the definition of a bialgebra in the sense of Drinfeld (D-bialgebra) related with any variety of algebras was stated. In particular, the associative and Jordan D-bialgebras were introduced, and an associative analogue of the Yang-Baxter equation was considered as well as the associative D-bialgebras related with the solutions to this equation. In the same papers, the associative algebras that admit a nontrivial structure of a D-bialgebra with cocommutative comultiplication on the center were also described. The comultiplication in an associative D-bialgebra is a derivation of an initial algebra into its tensor square considered as a bimodule over the initial algebra. These bialgebras were introduced in [4] and studied in [5]. The paper [5] is devoted to some properties of solutions to an associative analogue of the Yang-Baxter equation and the balanced bialgebras (i.e., D-bialgebras). The associative classical Yang-Baxter equations with parameters were considered in [6]. A class of Jordan D-bialgebras related to the “Jordan analogue” of the Yang-Baxter equation was introduced in [7], where it was proved that every finite-dimensional Jordan D-bialgebra, semisimple as an algebra, belongs to this class.

A so-called Manin triple may be associated with every Lie, associative, or Jordan bialgebra. In [8], the Manin triples for the associative algebras served as a tool for the study of the solutions to the Yang-Baxter equation.

Alternative D-bialgebras and their connection with the alternative Yang-Baxter equation were under study in [9]. In particular, the alternative D-bialgebra structures on Cayley-Dickson matrix algebra were described. Some connection of Jordan D-bialgebras with Lie bialgebras was revealed in [2]. It was shown in particular that under some natural restrictions a Jordan algebra J admits a nontrivial structure of a Jordan D-bialgebra if the Lie algebra $L(J)$ obtained by the Kantor-Koecher-Tits (KKT) construction from J admits the structure of a Lie bialgebra. Given an associative D-bialgebra (A, Δ) and its adjoint Jordan D-bialgebra $(A^{(+)}, \Delta^{(+)})$, $L(A^{(+)})$ may be equipped with the structure of a Lie bialgebra which is connected in some sense with $(A^{(+)}, \Delta^{(+)})$ [2]. In the present paper, we prove an analogue to this result in the case when A is a Cayley-Dickson matrix algebra and (A, Δ) is an alternative D-bialgebra. At the same time,

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we construct an example of alternative D-bialgebra (A, Δ) for which the structure of an adjoint Jordan D-bialgebra on $(A^{(+)}, \Delta^{(+)})$ cannot be extended to the structure of a Lie bialgebra on $L(A^{(+)})$.

Functional solutions to the classical Yang-Baxter equation on simple Lie algebras were constructed in [11]. In [12], using the ideas from [11], it was obtained an explicit description of Lie bialgebra structures on simple complex Lie algebras.

Malcev algebras were introduced by A.I. Malcev [13] as tangent algebras for local analytic Moufang loops. The class of Malcev algebras generalizes the class of Lie algebras and has a well developed theory [18].

An important example of a non-Lie Malcev algebra is the vector space of zero trace elements of a Caley-Dickson algebra with the commutator bracket multiplication [14, 15]. In [16] some properties of Malcev bialgebras were studied. In particular, there were found conditions for a Malcev algebra with a comultiplication to be a Malcev bialgebra.

In this work, we consider an analogue of the classical Yang-Baxter equation on Malcev algebras. In particular, it is shown that any solution to this equation induces a structure of a Malcev bialgebra. Also, we describe all structures of a Malcev bialgebra on the simple non-Lie complex Malcev algebra.

In order to perform vast volume of routine computations we used the Groups, Algorithms, Programming System (GAP).

§1. Definitions and Preliminaries

Given vector spaces V and U over a field F , denote by $V \otimes U$ its tensor product over F . Define the linear mapping τ on V by $\tau(\sum_i a_i \otimes b_i) = \sum_i b_i \otimes a_i$. Define the linear mapping ξ on $V \otimes V \otimes V$ by $\xi(\sum_i a_i \otimes b_i \otimes c_i) = \sum_i b_i \otimes c_i \otimes a_i$. Denote by V^* the dual space of V . Given $f \in V$ and $v \in V$, the symbol $\langle f, v \rangle$ denotes the linear functional f evaluated at v (i.e., $\langle f, a \rangle = f(a)$).

Definition. A pair (A, Δ) , where A is a vector space over F and $\Delta : A \rightarrow A \otimes A$ is a linear mapping, is called a *coalgebra*, while Δ is a *comultiplication*.

Given $a \in A$, put $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$.

Define some multiplication on A^* by

$$\langle fg, a \rangle = \sum_a \langle f, a_{(1)} \rangle \langle g, a_{(2)} \rangle,$$

where $f, g \in A^*$, $a \in A$ and $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$. The algebra obtained is *the dual algebra* of the coalgebra (A, Δ) .

The dual algebra A^* of (A, Δ) gives rise to the following bimodule actions on A :

$$f \rightharpoonup a = \sum a_{(1)} \langle f, a_{(2)} \rangle \text{ and } a \leftharpoonup f = \sum \langle f, a_{(1)} \rangle a_{(2)},$$

where $a \in A$, $f \in A^*$ and $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$.

The following definition of a coalgebra related to some variety of algebras was given in [20].

Definition. Let \mathcal{M} be an arbitrary variety of algebras. The pair (A, Δ) is called a \mathcal{M} -coalgebra if A^* belongs to \mathcal{M} .

Let A be an arbitrary algebra with a comultiplication Δ , and let A^* be the dual algebra for (A, Δ) . Then A induces the bimodule action on A^* by the formulas

$$\langle f \leftharpoonup a, b \rangle = \langle f, ab \rangle \text{ and } \langle b \rightharpoonup f, a \rangle = \langle f, ab \rangle,$$

where $a, b \in A$, $f \in A^*$.

Consider the space $D(A) = A \oplus A$ and equip it with the multiplication by putting

$$(a + f)(b + g) = (ab + f \rightharpoonup b + a \leftharpoonup g) + (fg + f \leftharpoonup b + a \rightharpoonup g).$$

Then $D(A)$ is an ordinary algebra over F , A and A^* are some subalgebras in $D(A)$. It is called *the Drinfeld double*.

Let Q be a bilinear form on $D(A)$ defined by

$$Q(a + f, b + g) = \langle g, a \rangle + \langle f, b \rangle$$

for all $a, b \in A$ and $f, g \in A^*$. It is easy to check that Q is a nondegenerate symmetric associative form, that is $Q(xy, z) = Q(x, yz)$.

Let us recall the definition of a Lie bialgebra. Let L be a Lie algebra with a comultiplication Δ . The pair (L, Δ) is called a Lie bialgebra if and only if (L, Δ) is a Lie coalgebra and Δ is a 1-cocycle, i.e., it satisfies

$$\Delta([a, b]) = \sum ([a_{(1)}, b] \otimes a_{(2)} + a_{(1)} \otimes [a_{(2)}, b]) + \sum ([a, b_{(1)}] \otimes b_{(2)} + b_{(1)} \otimes [a, b_{(2)}])$$

for all $a, b \in L$.

In [1], it was proved that the pair (L, Δ) is a Lie bialgebra if and only if its Drinfeld double $D(L)$ is a Lie algebra. This observation inspired the following definition [2].

Definition. Let \mathcal{M} be an arbitrary variety of algebras and let A be an algebra from \mathcal{M} with a comultiplication Δ . The pair (A, Δ) is called *an \mathcal{M} -bialgebra in the sense of Drinfeld* if its Drinfeld double $D(A)$ belongs to \mathcal{M} .

Note that this definition corresponds with the definition of coalgebra given in [20].

There is an important type of Lie bialgebras called coboundary bialgebras. Namely, let L be a Lie algebra and $R = \sum_i a_i \otimes b_i$ from $(id - \tau)(L \otimes L)$, that is, $\tau(R) = -R$. Define a comultiplication Δ_R on L by

$$\Delta_R(a) = \sum_i [a_i, a] \otimes b_i - a_i \otimes [a, b_i]$$

for all $a \in L$. It is easy to see that Δ_R is a 1-cocycle. In [11] it was proved that (L, Δ) is a Lie coalgebra if and only if the element

$$C_L(R) = [R_{12}, R_{13}] + [R_{12}, R_{23}] + [R_{13}, R_{23}]$$

is L -invariant. Here $[R_{12}, R_{13}] = \sum_{ij} [a_i, a_j] \otimes b_i \otimes b_j$, $[R_{12}, R_{23}] = \sum_{ij} b_i \otimes [a_i, a_j] \otimes b_j$, and $[R_{13}, R_{23}] = \sum_{ij} a_i \otimes a_j \otimes [b_i, b_j]$. In particular, if

$$[R_{12}, R_{13}] + [R_{12}, R_{23}] + [R_{13}, R_{23}] = 0, \tag{1}$$

then the pair (L, Δ_R) is a Lie bialgebra. In this case, we say that (L, Δ_R) is a triangular Lie bialgebra. The equation (1) is called *the classical Yang-Baxter equation*.

Let B be an arbitrary algebra and $r = \sum_i a_i \otimes b_i \in B \otimes B$. Then the equation

$$C_B(r) = r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = 0 \tag{2}$$

is called the classical Yang-Baxter equation on B . Here the subscripts specify the way of embedding $B \otimes B$ into $B \otimes B \otimes B$, that is, $r_{12} = \sum_i a_i \otimes b_i \otimes 1$, $r_{13} = \sum_i a_i \otimes 1 \otimes b_i$, $r_{23} = \sum_i 1 \otimes a_i \otimes b_i$. Note that $C_B(r)$ is well defined even if B is non-unital. This equation for different varieties of algebras were considered in [3, 5, 7, 6, 9]. Usually, antisymmetric solutions ($\tau(r) = -r$) to the equation (2) are considered.

An element $r = \sum_i a_i \otimes b_i \in B \otimes B$ induces a comultiplication Δ_r on B :

$$\Delta_r(a) = \sum a_i a \otimes b_i - a_i \otimes a b_i$$

for all $a \in B$.

We will need the following

Lemma 1. Let B be an arbitrary anticommutative finite-dimensional algebra over a field F , a_1, \dots, a_n be a basis of B , and γ_{ij}^k be the structure constants of B with respect to a_1, \dots, a_n , i.e., $a_i a_j = \sum_k \gamma_{ij}^k a_k$. Suppose that for an element $r = \sum_{ij} \alpha_{ij} a_i \otimes a_j \in B \otimes B$ and the dual algebra B^* of coalgebra $(B, -\Delta_r)$ the mapping $\phi : B^* \rightarrow B$ defined by $\phi(f) = \sum_{ij} \alpha_{ij} f(a_j) a_i$ is a homomorphism of algebras. Then the following equations hold:

$$(\Lambda^\top \Gamma_k \Lambda)_{sn} + (\Lambda \Gamma_s \Lambda)_{kn} + (\Lambda \Gamma_n \Lambda^\top)_{ks} = 0, \quad (3)$$

$$\det(\Lambda) \left(\sum_l 2(\Lambda \Gamma_l)_{kl} + (\Lambda^\top \Gamma_k)_{ll} \right) = 0, \quad (4)$$

where $\Lambda = (\alpha_{ij})_{i,j=1\dots n}$, $\Gamma_k = (\gamma_{ij}^k)_{i,j=1\dots n}$, and Λ^\top is the matrix transpose to Λ .

Proof. Let $b_k = \sum_i \alpha_{ki} a_i$. Then $r = \sum_i a_i \otimes b_i$. Since the mapping ϕ is a homomorphism, for all $f, g \in A^*$ we have

$$\begin{aligned} \sum_{i,j} f(a_i) g(a_j) b_i b_j &= \sum_i f g(a_i) b_i = \sum_i \langle f \otimes g, -\Delta_r(a_i) \rangle b_i = \\ &= - \left(\sum_{ij} f(a_j a_i) g(b_j) b_i - \sum_{ij} f(a_j) g(a_i b_j) b_i \right). \end{aligned}$$

Thus r satisfies the following condition:

$$\sum_{ij} a_i a_j \otimes b_i \otimes b_j - a_i \otimes a_j b_i \otimes b_j + a_i \otimes a_j \otimes b_i b_j = 0, \quad (5)$$

that is, r is a solution to the classical Yang-Baxter equation. Rewriting the elements b_i in terms of the elements a_i , one can get

$$\sum_{i,j,n,s,k} \gamma_{ij}^k \alpha_{is} \alpha_{jn} a_k \otimes a_s \otimes a_n - \gamma_{js}^k \alpha_{is} \alpha_{jn} a_i \otimes a_k \otimes a_n + \gamma_{ns}^k \alpha_{in} \alpha_{js} a_i \otimes a_j \otimes a_k = 0.$$

Changing corresponding indices in the second and third summands and using the skew-symmetry of Γ_i we conclude

$$\sum_{i,j,n,s,k} (\gamma_{ij}^k \alpha_{jn}) \alpha_{is} a_k \otimes a_s \otimes a_n + (\alpha_{ki} \gamma_{ij}^s) \alpha_{jn} a_k \otimes a_s \otimes a_n + (\alpha_{ki} \gamma_{ij}^n) \alpha_{sj} a_k \otimes a_s \otimes a_n = 0.$$

Therefore, for all k, s, n

$$\sum_{ij} (\gamma_{ij}^k \alpha_{jn}) \alpha_{is} + (\alpha_{ki} \gamma_{ij}^s) \alpha_{jn} + (\alpha_{ki} \gamma_{ij}^n) \alpha_{sj} = 0$$

Changing indices i and j in the first summand we obtain

$$\sum_{ij} (\gamma_{ji}^k \alpha_{in}) \alpha_{js} + (\alpha_{ki} \gamma_{ij}^s) \alpha_{jn} + (\alpha_{ki} \gamma_{ij}^n) \alpha_{sj} = 0$$

Hence,

$$\sum_j (\Gamma_k \Lambda)_{jn} \alpha_{js} + (\Lambda \Gamma_s)_{kj} \alpha_{jn} + (\Lambda \Gamma_n)_{kj} \alpha_{sj} = 0. \quad (6)$$

One can rewrite this equation in the following form:

$$(\Lambda^\top \Gamma_k \Lambda)_{sn} + (\Lambda \Gamma_s \Lambda)_{kn} + (\Lambda \Gamma_n \Lambda^\top)_{ks} = 0.$$

This proves the first statement (3).

Let $\Lambda^* = (\alpha_{ij}^*)$ be the cofactor matrix of Λ . Then $\Lambda \Lambda^{*\top} = \det(\Lambda) E$, where E is the identity matrix. Multiplying the equation (6) on α_{ln}^* and summing up over n we get

$$\sum_{jn} (\Gamma_k \Lambda)_{jn} \alpha_{js} \alpha_{ln}^* + (\Lambda \Gamma_s)_{kj} \alpha_{jn} \alpha_{ln}^* + (\Lambda \Gamma_n)_{kj} \alpha_{sj} \alpha_{ln}^* = 0.$$

Therefore,

$$\sum_{jn} ((\Gamma_k \Lambda)_{jn} \alpha_{js} \alpha_{ln}^* + (\Lambda \Gamma_n)_{kj} \alpha_{sj} \alpha_{ln}^*) + (\Lambda \Gamma_s)_{kl} \det(\Lambda) = 0.$$

The last equation can be rewritten in the form

$$\sum_n ((\Lambda^\top \Gamma_k \Lambda)_{sn} \alpha_{ln}^* + (\Lambda \Gamma_n \Lambda^\top)_{ks} \alpha_{ln}^*) + (\Lambda \Gamma_s)_{kl} \det(\Lambda) = 0.$$

Thus

$$\sum_n (\Lambda \Gamma_n \Lambda^\top)_{ks} \alpha_{ln}^* + (\Lambda^\top \Gamma_k \Lambda \Lambda^{*\top})_{sl} + (\Lambda \Gamma_s)_{kl} \det(\Lambda) = 0.$$

Hence,

$$\sum_n (\Lambda \Gamma_n \Lambda^\top)_{ks} \alpha_{ln}^* + (\Lambda^\top \Gamma_k)_{sl} \det(\Lambda) + (\Lambda \Gamma_s)_{kl} \det(\Lambda) = 0.$$

For $s = l$ we have

$$\sum_n (\Lambda \Gamma_n \Lambda^\top)_{kl} \alpha_{ln}^* + (\Lambda^\top \Gamma_k)_{ll} \det(\Lambda) + (\Lambda \Gamma_l)_{kl} \det(\Lambda) = 0.$$

Summing up over l we obtain

$$\sum_l \sum_n (\Lambda \Gamma_n \Lambda^\top)_{kl} \alpha_{ln}^* + \sum_l (\Lambda^\top \Gamma_k)_{ll} \det(\Lambda) + \sum_l (\Lambda \Gamma_l)_{kl} \det(\Lambda) = 0.$$

Then

$$\sum_n (\Lambda \Gamma_n \Lambda^\top \Lambda^*)_{kn} + \sum_l (\Lambda^\top \Gamma_k)_{ll} \det(\Lambda) + \sum_l (\Lambda \Gamma_l)_{kl} \det(\Lambda) = 0.$$

Therefore,

$$\sum_n (\Lambda \Gamma_n \det(\Lambda))_{kn} + \sum_l (\Lambda^\top \Gamma_k)_{ll} \det(\Lambda) + \sum_l (\Lambda \Gamma_l)_{kl} \det(\Lambda) = 0.$$

Finally, changing indexes n on l in the first sum, we obtain

$$\det(\Lambda) \left(\sum_l 2(\Lambda \Gamma_l)_{kl} + (\Lambda^\top \Gamma_k)_{ll} \right) = 0.$$

□

The following lemma was originally proved in [7]. Here we state its proof in order to complete the exposition.

Lemma 2. Let B be a finite-dimensional simple algebra over a field F with a nontrivial comultiplication Δ , $D(B) = B \oplus B^*$ be the Drinfeld double of the coalgebra (B, Δ) . Suppose that U is a nonzero ideal in $D(B)$ and

$$V = \{a \in B \mid a + f \in U \text{ for some } f \in B^*\}.$$

Then the dimensions of B and U are equal, the pair (V, Δ) is a subbialgebra of (B, Δ) , and $V^\perp U = UV^\perp = 0$, where V^\perp is the orthogonal complement of V in B^* .

PROOF. Let N be a B -subbimodule in B^* and N^\perp be the orthogonal complement of N in B with respect to the form Q . Then N^\perp is an ideal in J . Since B is a simple algebra, then either $N^\perp = 0$ or $N^\perp = B^*$. Therefore B^* is an irreducible B -bimodule.

Consider the vector space $W = \{f \in B^* \mid f + a \in U \text{ for some } a \in B\}$. Since $D(B) = B \oplus B^*$, then $W \neq 0$. Take $f \in W$ and $b \in B$. Then for some a from B we have $f + a \in U$. Since $(f + a)b = f \rightharpoonup b + ab + f \leftharpoonup b \in U$, then $f \leftharpoonup b \in W$. Similarly, $b \rightharpoonup f \in W$. Therefore W is a B -subbimodule in B^* and $W = B^*$. Hence, $\dim_F U \geq \dim_F B^*$. Note that $U \cap B = 0$ since B is a simple algebra. Therefore, $\dim_F(U + B) = \dim_F D(B)$ and $\dim_F U = \dim_F B^* = \dim_F B$.

Now, let us show that the pair (V, Δ) is a subbialgebra in (B, Δ) . For this, take $a \in V$ and $g \in B^*$. Then, for some $f \in B^*$ we have $a + f \in U$. Hence

$$(a + f)g = fg + a \rightharpoonup g + a \leftharpoonup g \in U.$$

Therefore $a \leftharpoonup g \in V$. Similarly, $g \rightharpoonup a \in V$ and (V, Δ) is a subcoalgebra in (B, Δ) . Consider $b \in V$, then

$$(a + f)b = f \leftharpoonup b + f \rightharpoonup b + ab \in U.$$

Hence V is a subbialgebra in B and the pair (V, Δ) is a subbialgebra in (B, Δ) .

Let b, n be arbitrary elements from B and V^\perp respectively. Then

$$(a + f)b = f \leftharpoonup b + f \rightharpoonup b + ab \in U.$$

Therefore $f \rightharpoonup b + ab \in V$. This implies that $Q(nf + n \leftharpoonup a, b) = Q(n, f \rightharpoonup b + ab) = 0$. Thus, we obtain $nf + n \leftharpoonup a = 0$. But any element $u \in U$ can be represented in the form $f + a$, where $f \in B^*$ and $a \in V$. Hence, $nu = n(a + f) = nf + n \leftharpoonup a + n \rightharpoonup a = 0$ and this proves that $V^\perp U = 0$. Similarly, $UV^\perp = 0$. □

§2. Coboundary and triangular Malcev bialgebras.

An anticommutative algebra is called a Malcev algebra if for all $x, y, z \in M$ the following equation holds:

$$J(x, y, xz) = J(x, y, z)x, \tag{7}$$

where $J(x, y, z) = (xy)z + (yz)x + (zx)y$ is the jacobian of elements x, y, z .

Note that if the characteristic of the field F is different from 2 then (7) is equivalent to the following equation:

$$((xy)z)t + ((yz)t)x + ((zt)x)y + ((tx)y)z = (xz)(yt). \quad (8)$$

For Malcev bialgebras, one can also consider the class of coboundary and triangular bialgebras.

Let M be a Malcev algebra, $r \in (id - \tau)(M \otimes M)$. Then r induces a comultiplication Δ_r on M defined by

$$\Delta_r(a) = [r, a] = \sum_i a_i a \otimes b_i - a_i \otimes ab_i$$

for all $a \in M$. In this work, we find necessary and sufficient conditions for the pair (M, Δ_r) to be a Malcev bialgebra.

In [16], the following statement was proved.

Theorem 1.(Vershinin). A Malcev algebra M with a multiplication μ and a comultiplication Δ is a Malcev bialgebra if and only if the dual algebra M^* is a Malcev algebra and the comultiplication satisfies

1. $\Delta((ab)c) + \Delta(bc)(a \otimes 1) + (b \otimes 1)\Delta(ac) = \sum a_{(1)}(bc) \otimes a_{(2)} + \sum a_{(1)}c \otimes a_{(2)}b + \sum a_{(1)} \otimes (a_{(2)}b)c + \sum ab_{(1)} \otimes b_{(2)}c - \sum b_{(1)} \otimes b_{(2)}(ac) + \sum (ab_{(1)})c \otimes b_{(2)} + \sum a(bc_{(1)}) \otimes c_{(2)} - \sum c_{(1)} \otimes (c_{(2)}a)b,$
2. $(1 \otimes \Delta)\Delta(ab) = (1 \otimes 1 \otimes a)((1 \otimes \Delta)\Delta(b)) + (\Delta \otimes 1)((a \otimes 1)\Delta(b)) - (1 \otimes \tau)((1 \otimes 1 \otimes a)(\Delta \otimes 1)\Delta(b)) + (1 \otimes \tau)((\Delta \otimes 1)\Delta(b))(a \otimes 1 \otimes 1) + ((\Delta \otimes 1)\Delta(a))(b \otimes 1 \otimes 1) + ((\Delta \otimes 1)\Delta(a))(1 \otimes 1 \otimes b) - (1 \otimes \tau)((\Delta \otimes 1)(\Delta(a)(b \otimes 1))) - (1 \otimes b \otimes 1)((1 \otimes \Delta)\Delta(a)) + (1 \otimes \tau)(1 \otimes 1 \otimes \mu)((1 \otimes \tau \otimes 1)(\Delta(b) \otimes \Delta(a))) + (1 \otimes 1 \otimes \mu)((1 \otimes \tau \otimes 1)(\Delta(a) \otimes \Delta(b))).$

In this work, we prove

Theorem 2. Let M be a Malcev algebra over a field characteristic not equal 2, $r \in (id - \tau)(M \otimes M)$. The pair (M, Δ_r) is a Malcev bialgebra if and only if for all $a, b \in M$

$$\begin{aligned} (C_M(r)(1 \otimes b \otimes 1))(1 \otimes a \otimes 1) - C_M(r)(ab \otimes 1 \otimes 1) - (C_M(r)(1 \otimes 1 \otimes a)), (1 \otimes 1 \otimes b) = \\ = C_M(r)(b \otimes 1 \otimes a) - C_M(r)(a \otimes b \otimes 1) \end{aligned} \quad (9)$$

or

$$C_M(r)(1 \otimes J_{b,a} \otimes 1 - 1 \otimes 1 \otimes J_{a,b}) = [C_M(r), ab] + [C_M(r), b](1 \otimes 1 \otimes a) - [C_M(r), a](1 \otimes b \otimes 1),$$

where the operator $J_{a,b}$ is defined by $cJ_{a,b} = J(c, a, b)$, and by $[C_M(r), a]$ we denote the action of M on $M \otimes M \otimes M$ defined by $[x \otimes y \otimes z, a] = xa \otimes y \otimes z + x \otimes ya \otimes z + x \otimes y \otimes za$.

PROOF. Let us prove the necessary condition. Since the pair (M, Δ_r) is a Malcev bialgebra then the second equation of the theorem 1 holds. We have:

$$(1 \otimes \Delta)\Delta_r(ab) =$$

$$\sum_{ij} a_i(ab) \otimes a_j b_i \otimes b_j - \sum_{ij} a_i(ab) \otimes a_j \otimes b_i b_j - \sum_{ij} a_i \otimes a_j((ab)b_i) \otimes b_j + \sum_{ij} a_i \otimes a_j \otimes ((ab)b_i)b_j.$$

$$(1 \otimes 1 \otimes a)((1 \otimes \Delta)\Delta(b)) =$$

$$\sum_{ij} a_i b \otimes a_j b_i \otimes ab_j - \sum_{ij} a_i b \otimes a_j \otimes a(b_i b_j) - \sum_{ij} a_i \otimes a_j(bb_i) \otimes ab_j + \sum_{ij} a_i \otimes a_j \otimes a((bb_i)b_j).$$

$$\begin{aligned}
& (\Delta \otimes 1)((a \otimes 1)\Delta(b)) = \\
& \sum_{ij} a_j(a(a_i b)) \otimes b_j \otimes b_i - \sum_{ij} a_j \otimes (a(a_i b))b_j \otimes b_i - \sum_{ij} a_j(aa_i) \otimes b_j \otimes bb_i + a_j \otimes (aa_i)b_j \otimes bb_i. \\
& -(1 \otimes \tau)((1 \otimes 1 \otimes a)(\Delta \otimes 1)\Delta(b)) = \\
& - \sum_{ij} a_j(a_i b) \otimes ab_i \otimes b_j + \sum_{ij} a_j a_i \otimes a(bb_i) \otimes b_j + \sum_{ij} a_j \otimes ab_i \otimes (a_i b)b_j - \sum_{ij} a_j \otimes a(bb_i) \otimes a_i b_j. \\
& (1 \otimes \tau)((\Delta \otimes 1)\Delta(b))(a \otimes 1 \otimes 1) = \\
& \sum_{ij} (a_j(a_i b))a \otimes b_i \otimes b_j - \sum_{ij} a_j a \otimes b_i \otimes (a_i b)b_j - \sum_{ij} (a_j a_i)a \otimes bb_i \otimes b_j + \sum_{ij} a_j a \otimes a_i b_j \otimes bb_i. \\
& ((\Delta \otimes 1)\Delta(a))(b \otimes 1 \otimes 1) = \\
& \sum_{ij} (a_j(a_i a))b \otimes b_j \otimes b_i - \sum_{ij} a_j b \otimes (a_i a)b_j \otimes b_i - \sum_{ij} (a_j a_i)b \otimes b_j \otimes ab_i + \sum_{ij} a_j b \otimes a_i b_j \otimes ab_i. \\
& ((\Delta \otimes 1)\Delta(a))(1 \otimes 1 \otimes b) = \\
& \sum_{ij} a_j(a_i a) \otimes b_j \otimes b_i b - \sum_{ij} a_j \otimes (a_i a_j)b_j \otimes b_i b - \sum_{ij} a_j a_i \otimes b_j \otimes (ab_i)b + \sum_{ij} a_j \otimes a_i b_j \otimes (ab_i)b. \\
& -(1 \otimes \tau)((\Delta \otimes 1)(\Delta(a)(b \otimes 1))) = \\
& - \sum_{ij} a_j((a_i a)b) \otimes b_i \otimes b_j + \sum_{ij} a_j \otimes b_i \otimes ((a_i a)b)b_j + \sum_{ij} a_j(a_i b) \otimes ab_i \otimes b_j - \sum_{ij} a_j \otimes ab_i \otimes (a_i b)b_j. \\
& -(1 \otimes b \otimes 1)((1 \otimes \Delta)\Delta(a)) = \\
& - \sum_{ij} a_i a \otimes b(a_j b_i) \otimes b_j + \sum_{ij} a_i a \otimes ba_j \otimes b_i b_j + \sum_{ij} a_i \otimes b(a_j(ab_i)) \otimes b_j - \sum_{ij} a_i \otimes ba_j \otimes (ab_i)b_j. \\
& (1 \otimes \tau)(1 \otimes 1 \otimes \mu)((1 \otimes \tau \otimes 1)(\Delta(b) \otimes \Delta(a))) = \\
& \sum_{ij} a_i b \otimes b_i b_j \otimes a_j a - \sum_{ij} a_i b \otimes b_i(ab_j) \otimes a_j - \sum_{ij} a_i \otimes (bb_i)b_j \otimes a_j a + \sum_{ij} a_i \otimes (bb_i)(ab_j) \otimes a_j. \\
& (1 \otimes 1 \otimes \mu)((1 \otimes \tau \otimes 1)(\Delta(a)\Delta(b))) = \\
& \sum_{ij} a_i a \otimes a_j b \otimes b_i b_j - \sum_{ij} a_i \otimes a_j b \otimes (ab_i)b_j - \sum_{ij} a_i a \otimes a_j \otimes b_i(bb_j) + \sum_{ij} a_i \otimes a_j \otimes (ab_i)(bb_j).
\end{aligned}$$

Inserting the expressions obtained into second equality of the theorem 1, using (8) and $\tau(r) = -r$, we conclude:

$$\begin{aligned}
& \sum_{ij} a_i(ab) \otimes a_j b_i \otimes b_j - \sum_{ij} a_i(ab) \otimes a_j \otimes b_i b_j - \sum_{ij} a_i \otimes ((b_i b_j)b)a \otimes a_j - \sum_{ij} a_i \otimes a_j \otimes ((b_i b_j)a)b = \\
& = - \sum_{ij} a_i b \otimes a_j \otimes a(b_i b_j) + \sum_{ij} a_i b \otimes a_j b_i \otimes ab_j - \sum_{ij} (a_j a_i)b \otimes b_j \otimes ab_i - \sum_{ij} (b_i b_j)(ab) \otimes a_j \otimes a_i +
\end{aligned}$$

$$\begin{aligned}
& \sum_{ij} a_j a_i \otimes b_j \otimes (b_i a) b + \sum_{ij} a_j \otimes a_i b_j \otimes (ab_i) b - \sum_{ij} b_i b_j \otimes (a_i b) a \otimes a_j - \sum_{ij} a_i \otimes (a_j b) a \otimes b_i b_j - \\
& - \sum_{ij} a_i a \otimes b(a_j b_i) \otimes b_j - \sum_{ij} (a_j a_i) a \otimes b b_i \otimes b_j + \sum_{ij} a_j a \otimes b b_i \otimes a_i b_j.
\end{aligned}$$

The last equality can be rewritten in the form

$$\begin{aligned}
& (C_M(r)(b \otimes 1 \otimes 1))(a \otimes 1 \otimes 1) - C_M(r)(ab \otimes 1 \otimes 1) - (C_M(r)(1 \otimes 1 \otimes a))(1 \otimes 1 \otimes b) = \\
& = C_M(r)(b \otimes 1 \otimes a) - C_M(r)(a \otimes b \otimes 1).
\end{aligned}$$

This proves the necessary condition.

Let us prove the sufficient condition. Let r satisfies (9). From the proof of the necessary condition it is clear that Δ_r satisfies the second condition of theorem 2. So we need to prove that M^* is a Malcev algebra and Δ_r satisfies the first condition of theorem 1.

In order to prove the first condition we need to check that the following equality holds:

$$\begin{aligned}
& \sum_i (a_i((ab)c) \otimes b_i - a_i \otimes ((ab)c)b_i) + \sum_i (a_i(bc) \otimes b_i a - a_i \otimes ((bc)b_i)a) + \sum_i (b(a_i(ac)) \otimes b_i - b a_i \otimes (ac)b_i) = \\
& = \sum_i ((a_i a)(bc) \otimes b_i - a_i(bc) \otimes ab_i) - \sum_i (a_i c \otimes (b_i a)b + a_i \otimes ((cb_i)a)b) + \sum_i ((a_i a)c \otimes b_i b - a_i c \otimes (ab_i)b) + \\
& + \sum_i ((a(a_i b))c \otimes b_i - (aa_i)c \otimes bb_i) + \sum_i (a_i a \otimes (b_i b)c - a_i \otimes ((ab_i)b)c) + \sum_i (a(a_i b) \otimes b_i c - aa_i \otimes (bb_i)c) + \\
& \sum_i (a(b(a_i c)) \otimes b_i - a(ba_i) \otimes cb_i) - \sum_i (a_i b \otimes b_i(ac) + a_i \otimes (bb_i)(ac)).
\end{aligned}$$

It holds due to (8).

Let us prove that the dual algebra M^* is a Malcev algebra.

For any $a \in M$ and $f, g, h, t \in M^*$ we have

$$\begin{aligned}
1. \langle ((fg)h)t, a \rangle &= \sum_{ijk} \langle f, a_k(a_j(a_i a)) \rangle \langle g, b_k \rangle \langle h, b_j \rangle \langle t, b_i \rangle - \sum_{ijk} \langle f, a_k \rangle \langle g, (a_j(a_i a))b_k \rangle \langle h, b_j \rangle \langle t, b_i \rangle - \\
& - \sum_{ijk} \langle f, a_k a_j \rangle \langle g, b_k \rangle \langle h, (a_i a)b_j \rangle \langle t, b_i \rangle + \sum_{ijk} \langle f, a_k \rangle \langle g, a_j b_k \rangle \langle h, (a_i a)b_j \rangle \langle t, b_i \rangle - \\
& - \sum_{ijk} \langle f, a_k(a_j a_i) \rangle \langle g, b_k \rangle \langle h, b_j \rangle \langle t, ab_i \rangle + \sum_{ijk} \langle f, a_k \rangle \langle g, (a_j a_i)b_k \rangle \langle h, b_j \rangle \langle t, ab_i \rangle + \\
& + \sum_{ijk} \langle f, a_k a_j \rangle \langle g, b_k \rangle \langle h, a_i b_j \rangle \langle t, ab_i \rangle - \sum_{ijk} \langle f, a_k \rangle \langle g, a_j b_k \rangle \langle h, a_i b_j \rangle \langle t, ab_i \rangle. \\
2. \langle ((gh)t)f, a \rangle &= \sum_{ijk} \langle f, b_i \rangle \langle g, a_k(a_j(a_i a)) \rangle \langle h, b_k \rangle \langle t, b_j \rangle - \sum_{ijk} \langle f, b_i \rangle \langle g, a_k \rangle \langle h, (a_j(a_i a))b_k \rangle \langle t, b_j \rangle - \\
& - \sum_{ijk} \langle f, b_i \rangle \langle g, a_k a_j \rangle \langle h, b_k \rangle \langle t, (a_i a)b_j \rangle + \sum_{ijk} \langle f, b_i \rangle \langle g, a_k \rangle \langle h, a_j b_k \rangle \langle t, (a_i a)b_j \rangle -
\end{aligned}$$

$$\begin{aligned}
& - \sum_{ijk} \langle f, ab_i \rangle \langle g, a_k(a_j a_i) \rangle \langle h, b_k \rangle \langle t, b_j \rangle + \sum_{ijk} \langle f, ab_i \rangle \langle g, a_k \rangle \langle h, (a_j a_i) b_k \rangle \langle t, b_j \rangle + \\
& + \sum_{ijk} \langle f, ab_i \rangle \langle g, a_k a_j \rangle \langle h, b_k \rangle \langle t, a_i b_j \rangle - \sum_{ijk} \langle f, ab_i \rangle \langle g, a_k \rangle \langle h, a_j b_k \rangle \langle t, a_i b_j \rangle.
\end{aligned}$$

$$\begin{aligned}
3. \langle ((ht)f)g, a \rangle &= \sum_{ijk} \langle f, b_j \rangle \langle g, b_i \rangle \langle h, a_k(a_j(a_i a)) \rangle \langle t, b_k \rangle - \sum_{ijk} \langle f, b_j \rangle \langle g, b_i \rangle \langle h, a_k \rangle \langle t, (a_j(a_i a)) b_k \rangle - \\
& - \sum_{ijk} \langle f, (a_i a) b_j \rangle \langle g, b_i \rangle \langle h, a_k a_j \rangle \langle t, b_k \rangle + \sum_{ijk} \langle f, (a_i a) b_j \rangle \langle g, b_i \rangle \langle h, a_k \rangle \langle t, a_j b_k \rangle - \\
& - \sum_{ijk} \langle f, b_j \rangle \langle g, ab_i \rangle \langle h, a_k(a_j a_i) \rangle \langle t, b_k \rangle + \sum_{ijk} \langle f, b_j \rangle \langle g, ab_i \rangle \langle h, a_k \rangle \langle t, (a_j a_i) b_k \rangle + \\
& + \sum_{ijk} \langle f, a_i b_j \rangle \langle g, ab_i \rangle \langle h, a_k a_j \rangle \langle t, b_k \rangle - \sum_{ijk} \langle f, a_i b_j \rangle \langle g, ab_i \rangle \langle h, a_k \rangle \langle t, a_j b_k \rangle.
\end{aligned}$$

$$\begin{aligned}
4. \langle ((tf)g)h, a \rangle &= \sum_{ijk} \langle f, b_k \rangle \langle g, b_j \rangle \langle h, b_i \rangle \langle t, a_k(a_j(a_i a)) \rangle - \sum_{ijk} \langle f, (a_j(a_i a)) b_k \rangle \langle g, b_j \rangle \langle h, b_i \rangle \langle t, a_k \rangle - \\
& - \sum_{ijk} \langle f, b_k \rangle \langle g, (a_i a) b_j \rangle \langle h, b_i \rangle \langle t, a_k a_j \rangle + \sum_{ijk} \langle f, a_j b_k \rangle \langle g, (a_i a) b_j \rangle \langle h, b_i \rangle \langle t, a_k \rangle - \\
& - \sum_{ijk} \langle f, b_k \rangle \langle g, b_j \rangle \langle h, ab_i \rangle \langle t, a_k(a_j a_i) \rangle + \sum_{ijk} \langle f, (a_j a_i) b_k \rangle \langle g, b_j \rangle \langle h, ab_i \rangle \langle t, a_k \rangle + \\
& + \sum_{ijk} \langle f, b_k \rangle \langle g, a_i b_j \rangle \langle h, ab_i \rangle \langle t, a_k a_j \rangle - \sum_{ijk} \langle f, a_j b_k \rangle \langle g, a_i b_j \rangle \langle h, ab_i \rangle \langle t, a_k \rangle.
\end{aligned}$$

$$\begin{aligned}
5. \langle (fh)(gt), a \rangle &= \sum_{ijk} \langle f, a_j(a_i a) \rangle \langle g, a_k b_i \rangle \langle h, b_j \rangle \langle t, b_k \rangle - \sum_{ijk} \langle f, a_j(a_i a) \rangle \langle g, a_k \rangle \langle h, b_j \rangle \langle t, b_i b_k \rangle - \\
& - \sum_{ijk} \langle f, a_j \rangle \langle g, a_k b_i \rangle \langle h, (a_i a) a_j \rangle \langle t, b_k \rangle + \sum_{ijk} \langle f, a_j \rangle \langle g, a_k \rangle \langle h, (a_i a) a_j \rangle \langle t, b_i b_k \rangle - \\
& - \sum_{ijk} \langle f, a_j a_i \rangle \langle g, a_k(ab_i) \rangle \langle h, b_j \rangle \langle t, b_k \rangle + \sum_{ijk} \langle f, a_j a_i \rangle \langle g, a_k \rangle \langle h, b_j \rangle \langle t, (ab_i) b_k \rangle + \\
& \sum_{ijk} \langle f, a_j \rangle \langle g, a_k(ab_i) \rangle \langle h, a_i b_j \rangle \langle t, b_k \rangle - \sum_{ijk} \langle f, a_j \rangle \langle g, a_k \rangle \langle h, a_i b_j \rangle \langle t, (ab_i) b_k \rangle.
\end{aligned}$$

In what follows we will use the following notation: an expression $x \otimes y \otimes z \otimes s^{i,j}$, $i = 1, \dots, 5$, $j = 1, \dots, 8$, means that $\langle f, x \rangle \langle g, y \rangle \langle h, z \rangle \langle t, s \rangle$ is equal to the j th summand of the i th equality (if $i < 5$) or to the negative j th summand of the 5th equality (if $i = 5$).

Denote by

$$\begin{aligned}
S(a, b) &= (C_M(r)(1 \otimes b \otimes 1))(1 \otimes a \otimes 1) - C_M(r)(ab \otimes 1 \otimes 1) - (C_M(r)(1 \otimes 1 \otimes a))(1 \otimes 1 \otimes b) - \\
& - C_M(r)(b \otimes 1 \otimes a) + C_M(r)(a \otimes b \otimes 1).
\end{aligned}$$

Let $Q = \sum_i (\xi^2(S(a, a_i))) \otimes b_i = 0$. Then by (2) $S(a, b) = 0$ for all $a, b \in M$. Consequently, $Q = 0$. Thus (here, expressions $p^{(i)}$, $q^{(i)}$ mean, that the elements p and q are equal)

$$\begin{aligned}
Q = & - \sum_{ijk} ((a_j a_k) a_i) a \otimes b_k \otimes b_j \otimes b_i - \sum_{ijk} (b_k a_i) a \otimes b_j \otimes a_j a_k \otimes b_i^{(1)} - \sum_{ijk} (b_j a_i) a \otimes a_j a_k \otimes b_k \otimes b_i^{(2)} - \\
& - \sum_{ijk} a_j a_k \otimes b_k \otimes b_j (a_i a) \otimes b_i^{1.3} - \sum_{ijk} b_k \otimes b_j \otimes (a_j a_k) (a_i a) \otimes b_i - \sum_{ijk} b_j \otimes a_j a_k \otimes b_k (a_i a) \otimes b_i^{1.4} + \\
& + \sum_{ijk} a_j a_k \otimes (b_k a) a_i \otimes b_j \otimes b_i^{5.5} + \sum_{ijk} b_k \otimes (b_j a) a_i \otimes a_j a_k \otimes b_i^{5.7} + \sum_{ijk} b_j \otimes ((a_j a_k) a) a_i \otimes b_k \otimes b_i - \\
& - \sum_{ijk} (a_j a_k) a_i \otimes b_k \otimes b_j a \otimes b_i^{4.6} - \sum_{ijk} b_k a_i \otimes b_j \otimes (a_j a_k) a \otimes b_i^{(3)} - \sum_{ijk} b_j a_i \otimes a_j a_k \otimes b_k a \otimes b_i^{4.8} + \\
& + \sum_{ijk} a_j a_k \otimes b_k a \otimes b_j a_i \otimes b_i^{3.7} + \sum_{ijk} b_k \otimes b_j a \otimes (a_j a_k) a_i \otimes b_i^{3.5} + \sum_{ijk} b_j \otimes (a_j a_k) a \otimes b_k a_i \otimes b_i^{(4)} = 0.
\end{aligned}$$

Let ς be the linear mapping of $M \otimes M \otimes M \otimes M$ defined by $\varsigma(x \otimes y \otimes z \otimes t) = y \otimes z \otimes t \otimes x$. Then

$$\begin{aligned}
\varsigma(Q) = & - \sum_{ijk} b_i \otimes ((a_j a_k) a_i) a \otimes b_k \otimes b_j - \sum_{ijk} b_i \otimes (b_k a_i) a \otimes b_j \otimes a_j a_k^{(5)} - \sum_{ijk} b_i \otimes (b_j a_i) a \otimes a_j a_k \otimes b_k^{(4)} - \\
& - \sum_{ijk} b_i \otimes a_j a_k \otimes b_k \otimes b_j (a_i a)^{2.3} - \sum_{ijk} b_i \otimes b_k \otimes b_j \otimes (a_j a_k) (a_i a) - \sum_{ijk} b_i \otimes b_j \otimes a_j a_k \otimes b_k (a_i a)^{2.4} + \\
& + \sum_{ijk} b_i \otimes a_j a_k \otimes (b_k a) a_i \otimes b_j^{5.3} + \sum_{ijk} b_i \otimes b_k \otimes (b_j a) a_i \otimes a_j a_k^{5.4} + \sum_{ijk} b_i \otimes b_j \otimes ((a_j a_k) a) a_i \otimes b_k - \\
& - \sum_{ijk} b_i \otimes (a_j a_k) a_i \otimes b_k \otimes b_j a^{1.6} - \sum_{ijk} b_i \otimes b_k a_i \otimes b_j \otimes (a_j a_k) a^{(6)} - \sum_{ijk} b_i \otimes b_j a_i \otimes a_j a_k \otimes b_k a^{1.8} + \\
& + \sum_{ijk} b_i \otimes a_j a_k \otimes b_k a \otimes b_j a_i^{4.7} + \sum_{ijk} b_i \otimes b_k \otimes b_j a \otimes (a_j a_k) a_i^{4.5} + \sum_{ijk} b_i \otimes b_j \otimes (a_j a_k) a \otimes b_k a_i^{(7)} = 0.
\end{aligned}$$

$$\begin{aligned}
\varsigma^2(Q) = & - \sum_{ijk} b_j \otimes b_i \otimes ((a_j a_k) a_i) a \otimes b_k - \sum_{ijk} a_j a_k \otimes b_i \otimes (b_k a_i) a \otimes b_j^{(3)} - \sum_{ijk} b_k \otimes b_i \otimes (b_j a_i) a \otimes a_j a_k^{(7)} - \\
& - \sum_{ijk} b_j (a_i a) \otimes b_i \otimes a_j a_k \otimes b_k^{3.3} - \sum_{ijk} (a_j a_k) (a_i a) \otimes b_i \otimes b_k \otimes b_j - \sum_{ijk} b_k (a_i a) \otimes b_i \otimes b_j \otimes a_j a_k^{3.4} + \\
& + \sum_{ijk} b_j \otimes b_i \otimes a_j a_k \otimes (b_k a) a_i^{5.8} + \sum_{ijk} a_j a_k \otimes b_i \otimes b_k \otimes (b_j a) a_i^{5.6} + \sum_{ijk} b_k \otimes b_i \otimes b_j \otimes ((a_j a_k) a) a_i - \\
& - \sum_{ijk} b_j a \otimes b_i \otimes (a_j a_k) a_i \otimes b_k^{2.6} - \sum_{ijk} (a_j a_k) a \otimes b_i \otimes b_k a_i \otimes b_j^{(1)} - \sum_{ijk} b_k a \otimes b_i \otimes b_j a_i \otimes a_j a_k^{2.8} + \\
& + \sum_{ijk} b_j a_i \otimes b_i \otimes a_j a_k \otimes b_k a^{1.7} + \sum_{ijk} (a_j a_k) a_i \otimes b_i \otimes b_k \otimes b_j a^{1.5} + \sum_{ijk} b_k a_i \otimes b_i \otimes b_j \otimes (a_j a_k) a^{(8)} = 0.
\end{aligned}$$

$$\begin{aligned}
\varsigma^3(Q) = & - \sum_{ijk} b_k \otimes b_j \otimes b_i \otimes ((a_j a_k) a_i) a - \sum_{ijk} b_j \otimes a_j a_k \otimes b_i \otimes (b_k a_i) a^{(6)} - \sum_{ijk} a_j a_k \otimes b_k \otimes b_i \otimes (b_j a_i) a^{(8)} - \\
& - \sum_{ijk} b_k \otimes b_j (a_i a) \otimes b_i \otimes a_j a_k^{4.3} - \sum_{ijk} b_j \otimes (a_j a_k) (a_i a) \otimes b_i \otimes b_k - \sum_{ijk} a_j a_k \otimes b_k (a_i a) \otimes b_i \otimes b_j^{4.4} + \\
& + \sum_{ijk} (b_k a) a_i \otimes b_j \otimes b_i \otimes a_j a_k^{5.2} + \sum_{ijk} (b_j a) a_i \otimes a_j a_k \otimes b_i \otimes b_k^{5.1} + \sum_{ijk} ((a_j a_k) a) a_i \otimes b_k \otimes b_i \otimes b_j - \\
& - \sum_{ijk} b_k \otimes b_j a \otimes b_i \otimes (a_j a_k) a_i^{3.6} - \sum_{ijk} b_j \otimes (a_j a_k) a \otimes b_i \otimes b_k a_i^{(5)} - \sum_{ijk} a_j a_k \otimes b_k a \otimes b_i \otimes b_j a_i^{3.8} + \\
& \sum_{ijk} b_k a \otimes b_j a_i \otimes b_i \otimes a_j a_k^{2.7} + \sum_{ijk} b_j a \otimes (a_j a_k) a_i \otimes b_i \otimes b_k^{2.5} + \sum_{ijk} (a_j a_k) a \otimes b_k a_i \otimes b_i \otimes b_j^{(2)} = 0.
\end{aligned}$$

Using (8) we get

$$\begin{aligned}
& - \sum_{ijk} ((a_j a_k) a_i) a \otimes b_k \otimes b_j \otimes b_i - \sum_{ijk} (a_j a_k) (a_i a) \otimes b_i \otimes b_k \otimes b_j + \sum_{ijk} ((a_j a_k) a) a_i \otimes b_k \otimes b_i \otimes b_j = \\
& = \sum_{ijk} ((a_i a) a_j) a_k \otimes b_k \otimes b_j \otimes b_i^{1.1} + \sum_{ijk} ((a a_j) a_k) a_i \otimes b_k \otimes b_j \otimes b_i^{4.2}
\end{aligned}$$

Similarly, one can transform the following elements

$$\begin{aligned}
& \sum_{ijk} b_j \otimes ((a_j a_k) a) a_i \otimes b_k \otimes b_i - \sum_{ijk} b_i \otimes ((a_j a_k) a_i) a \otimes b_k \otimes b_j - \sum_{ijk} b_j \otimes (a_j a_k) (a_i a) \otimes b_i \otimes b_k = \\
& = \sum_{ijk} b_i \otimes a_k (a_j (a_i a)) \otimes b_k \otimes b_j \otimes b_i^{2.1} - \sum_{ijk} a_k \otimes (a_j (a_i a)) b_k \otimes b_j \otimes b_i^{1.2}, \\
& - \sum_{ijk} b_k \otimes b_j \otimes (a_j a_k) (a_i a) \otimes b_i + \sum_{ijk} b_i \otimes b_j \otimes ((a_j a_k) a) a_i \otimes b_k - \sum_{ijk} b_j \otimes b_i \otimes ((a_j a_k) a_i) a \otimes b_k = \\
& = \sum_{ijk} b_j \otimes b_i \otimes a_k (a_j (a_i a)) \otimes b_k^{3.1} - \sum_{ijk} b_i \otimes a_k \otimes (a_j (a_i a)) b_k \otimes b_j^{2.2} \text{ and} \\
& - \sum_{ijk} b_i \otimes b_k \otimes b_j \otimes (a_j a_k) (a_i a) + \sum_{ijk} b_k \otimes b_i \otimes b_j \otimes ((a_j a_k) a) a_i - \sum_{ijk} b_k \otimes b_j \otimes b_i \otimes ((a_j a_k) a_i) a = \\
& = \sum_{ijk} b_k \otimes b_j \otimes b_i \otimes a_k (a_j (a_i a))^{4.1} - \sum_{ijk} b_j \otimes b_i \otimes a_k \otimes (a_j (a_i a)) b_k^{3.2}.
\end{aligned}$$

Summing up $(id + \varsigma + \varsigma^2 + \varsigma^3)Q$, taking into account the last equalities, and acting on the sum by $f \otimes g \otimes h \otimes t$, one can finally get

$$\langle ((fg)h)t, a \rangle + \langle ((gh)t)f, a \rangle + \langle ((ht)f)g, a \rangle + \langle ((tf)g)h, a \rangle - \langle (fh)(gt), a \rangle = 0.$$

Thus, the dual algebra M^* is a Malcev algebra. \square

Corollary 1. Let M be a Malcev algebra over a field of characteristic not 2, an element $r \in (id - \tau)(M \otimes M)$ be a solution to the classical Yang-Baxter equation on M . Then the pair (M, Δ_r) is a Malcev bialgebra.

§3. Structures of a Malcev bialgebra on the non-Lie simple Malcev algebra - preliminary results.

Let F be an algebraically closed field of characteristic different from 2 and 3. Then, up to an isomorphism, there is only one non-Lie simple Malcev algebra \mathbb{M} over F ([17]). The dimension of \mathbb{M} is equal to seven, and it is convenient to consider a base h, x, x', y, y', z, z' of \mathbb{M} with the following multiplication table:

$$\begin{aligned} hx &= 2x, \quad hy = 2y, \quad hz = 2z, \\ hx' &= -2x', \quad hy' = -2y', \quad hz' = -2z', \\ xx' &= yy' = zz' = h, \\ xy &= 2z', \quad yz = 2x', \quad zx = 2y', \\ x'y' &= -2z, \quad y'z' = -2x, \quad z'x' = -2y. \end{aligned}$$

The remaining products are zero. Such a basis is called a standard basis.

The algebra \mathbb{M} can be constructed in the following way. Let \mathcal{C} be the matrix Cayley-Dickson algebra with a multiplication $x \cdot y$. Then $\mathcal{C} = F_2 + vF_2$, where F_2 is the algebra of 2×2 matrices. Define a new multiplication in \mathcal{C} : $xy = \frac{1}{2}(x \cdot y - y \cdot x)$. Then the vector space \mathcal{C} with this multiplication turns into a Malcev algebra denoted by $\mathcal{C}^{(-)}$. The set $F \cdot 1$, where 1 is the unit of \mathcal{C} , is the center of $\mathcal{C}^{(-)}$, and the quotient algebra $\mathcal{C}^{(-)}/F \cdot 1$ is isomorphic to \mathbb{M} .

From the multiplication table one can see that the space $M(4)$ with the basis h, x, y', z is a four-dimensional subalgebra in \mathbb{M} . This algebra first appeared in [19].

Lemma 3. Suppose B is a subalgebra in \mathbb{M} of dimension 4. Then one can choose a standard basis of \mathbb{M} in such a way that $B = M(4)$.

PROOF. The algebra \mathcal{C} can be represented in the following way: $\mathcal{C} = F \cdot 1 + \mathbb{M}$. The multiplication in \mathcal{C} is given by the formula ([18])

$$a \cdot b = -(a, b)1 + ab,$$

where $a, b \in \mathbb{M}$, (\cdot, \cdot) is a symmetric non-degenerate associative bilinear form on \mathbb{M} and ab is the antisymmetric multiplication in \mathbb{M} . It is clear that $B' = B + F \cdot 1$ is a five-dimensional subalgebra in \mathcal{C} , and B is isomorphic to the quotient algebra $B'^{(-)}/F \cdot 1$. Since B' is a non-associative subalgebra in \mathcal{C} , then B is non-Lie. In [21] it was proved that over an algebraically closed field of characteristic different from 2 any four-dimensional non-Lie subalgebra of \mathbb{M} is isomorphic to $M(4)$. Thus, we can choose a basis of B with the same multiplication table for h, x, y', z . So, we can suppose that h, x, y', z is the base of B . Let L be a subspace generated by x, y', z . Then $(h, L) = (L, L) = 0$ and it can be assumed that $(h, h) = -\frac{1}{4}$. Then the element $h + \frac{1}{2} \cdot 1$ is an idempotent, so the space B_1 with the base $h + \frac{1}{2} \cdot 1, x, y', z$ is a subalgebra of \mathcal{C} , and B is isomorphic to the quotient algebra $B_1^{(-)}/F \cdot 1$. It is easy to see that $B_1 = F \cdot (h + \frac{1}{2} \cdot 1) + L$ and L is a nilpotent ideal of B_1 . We can assume that $h + \frac{1}{2} \cdot 1 = e_{11}$ ([9, Lemma 5]) and B_1 has one of the following bases: $e_{11}, e_{12}, ve_{11}, ve_{12}$ or $e_{11}, ve_{21}, ve_{22}, e_{21}$, where e_{ij} is the matrix unit of F_2 . Then the images of

$$-4e_{11}, 2ve_{11}, 2ve_{22}, 2e_{21}, 2e_{12}, 2ve_{12}, -2ve_{21}$$

under the canonical homomorphism $\mathcal{C}^{(-)} \mapsto \mathcal{C}^{(-)}/F \cdot 1$ form a standard basis of \mathbb{M} . It is clear that in this basis $B = M(4)$. \square

Suppose (\mathbb{M}, Δ) is a Malcev bialgebra. Consider the Drinfeld double $D(\mathbb{M})$. Obviously, $D(\mathbb{M})$ is not a simple algebra. Therefore $D(\mathbb{M})$ is either a semisimple Malcev algebra or it possesses a non-zero radical(=solvable radical) R . In the following two sections we consider each of these cases separately.

§4. The case of non-zero radical.

Lemma 4. Suppose the radical R of the Drinfeld double $D(\mathbb{M})$ is nonzero. Then $R = R^\perp$, $R^2 = 0$ and $D(\mathbb{M}) = \mathbb{M} + R$ (semidirect sum).

PROOF. By Lemma 1 $\dim R = \dim \mathbb{M}$. Since \mathbb{M} is a simple algebra and \mathbb{M} is not an ideal of $D(\mathbb{M})$, we have $\mathbb{M} \cap R = 0$ and $D(\mathbb{M}) = \mathbb{M} + R$. Then, $\mathbb{M} \cong D(\mathbb{M})/R$.

Let R^\perp be the orthogonal complement of R in $D(\mathbb{M})$ with respect to Q . It is straightforward to see that R^\perp is an ideal of $D(\mathbb{M})$, so $\dim R^\perp = \dim \mathbb{M}$ by Lemma 2. Therefore, $D(\mathbb{M}) = \mathbb{M} + R^\perp$ and \mathbb{M} is isomorphic to the quotient algebra $D(\mathbb{M})/R^\perp$. Let us consider that $R \cap R^\perp = 0$. Then $D(\mathbb{M}) = R \oplus R^\perp$ (the direct sum of algebras) and R is isomorphic to the quotient algebra $D(\mathbb{M})/R^\perp$. It shows that the algebras \mathbb{M} and R are isomorphic what contradicts the simplicity of \mathbb{M} .

Thus $R \cap R^\perp = R$ and $R = R^\perp$. Since $Q(R^2, D(\mathbb{M})) = Q(R, RD(\mathbb{M})) = 0$, we finally get $R^2 = 0$. \square

Theorem 3. Let a pair (\mathbb{M}, Δ) be a Malcev bialgebra, and let the radical R of the Drinfeld double $D(\mathbb{M})$ is non-zero. Then there exists an element r from $(id - \tau)(\mathbb{M} \otimes \mathbb{M})$ such that $\Delta = \Delta_r$ and r is a solution to the classical Yang-Baxter equation on \mathbb{M} .

PROOF. Consider $D(\mathbb{M})$. By Lemma 4 $D(\mathbb{M}) = \mathbb{M} + R$, so for every $f \in \mathbb{M}^*$ there exists an element $a \in \mathbb{M}$ such that $f = a + u$, where $u \in R$. Define a mapping $\phi : \mathbb{M}^* \rightarrow \mathbb{M}$ by $\phi(f) = a$. The mapping ϕ is a well-defined homomorphism of algebras. Since \mathbb{M}^* is the dual space to \mathbb{M} , one can find an element $r = \sum_i a_i \otimes b_i \in \mathbb{M} \otimes \mathbb{M}$, such that $\phi(f) = \sum_i f(b_i) a_i$.

Take $f, g \in \mathbb{M}^*$. Since $R^\perp = R$, then $Q(f - \phi(f), g - \phi(g)) = 0$. Consequently,

$$\sum_i f(b_i) g(a_i) + f(a_i) g(b_i) = 0.$$

Therefore,

$$\sum_i \langle f \otimes g, \sum_i a_i \otimes b_i + b_i \otimes a_i \rangle = 0.$$

Finally, $\tau(r) = -r$.

Further, since $R^2 = 0$, we get $(f - \sum_i f(a_i) b_i)(g - \sum_i g(a_i) b_i) = 0$. Hence,

$$fg - \sum_i g(b_i) f \lrcorner a_i - \sum_i f(b_i) a_i \rightharpoonup g = 0.$$

Thus, for all $a \in \mathbb{M}$

$$fg(a) = \sum_i \langle f \otimes g, a_i a \otimes b_i + b_i \otimes a a_i \rangle.$$

Therefore, $\Delta(a) = \sum_i (a_i a \otimes b_i + b_i \otimes a a_i)$ for all $a \in \mathbb{M}$. Since $\tau(r) = -r$, then $\Delta(a) = [r, a] = \sum_i a_i a \otimes b_i - a_i \otimes a b_i$. In other words, $\Delta = \Delta_r$.

Since ϕ is a homomorphism, we have

$$\sum_i f(b_i) g(b_j) a_i a_j = \sum_i f g(b_i) a_i = \sum_{ij} f(a_j b_i) g(b_j) a_i - \sum_{ij} f(a_j) g(b_i b_j) a_i.$$

It follows that $\langle f \otimes g \otimes h, C_{\mathbb{M}}(r) \rangle = 0$ for all $f, g, h \in \mathbb{M}^*$. Therefore $C_{\mathbb{M}}(r) = 0$. \square

Theorem 4. Let r be an antisymmetric solution to the classical Yang-Baxter equation on M . Then the pair (\mathbb{M}, Δ_r) is a Malcev bialgebra. Moreover, the radical R of the Drinfeld double $D(\mathbb{M})$ is nonzero.

PROOF. The first statement is valid due to the Theorem 2. Consider a mapping $\phi : \mathbb{M}^* \rightarrow \mathbb{M}$ defined by $\phi(f) = \sum_i f(b_i) a_i$. It is easy to see that ϕ is a homomorphism of algebras. Consider a set $S = \{f - \phi(f) \mid f \in \mathbb{M}^*\}$. For all $a \in \mathbb{M}, f, g \in \mathbb{M}^*$ we have

$$(a + g)(f - \phi(f)) = -a\phi(f) + a \leftarrow f - g \rightarrow \phi(f) + fg + a \rightarrow f - g \leftarrow \phi(f).$$

Since the action of ϕ on $fg + a \rightarrow f - g \leftarrow \phi(f)$ leads to $a\phi(f) - a \leftarrow f + g \rightarrow \phi(f)$, the right-hand side belongs to S , i.e., S is an ideal in $D(\mathbb{M})$.

For all $f, g \in \mathbb{M}^*$ we have

$$Q(f - \phi(f), g - \phi(g)) = -f(\phi(g)) - g(\phi(f)) = \langle f \otimes g, \sum_i a_i \otimes b_i + b_i \otimes a_i \rangle = 0.$$

Using the associativity of Q we obtain

$$Q(a + h, (f - \phi(f))(g - \phi(g))) = Q((a + f)(f - \phi(f)), g - \phi(g)) = 0$$

for all $a + h \in D(\mathbb{M})$, so $R^2 = 0$. Consequently, $S \subseteq R$ and so $R \neq 0$. \square

Thus, in order to describe all Malcev bialgebra structures on \mathbb{M} one should find all antisymmetric solutions of the classical Yang-Baxter equation (2). On the other hand, these solutions ([11, 22, 9]) are in one to one correspondence with pairs (B, ω) , where B is a subalgebra of \mathbb{M} , ω is a non-degenerate skew-symmetric bilinear form satisfying

$$\omega(xy, z) + \omega(yz, x) + \omega(zx, y) = 0$$

for all $x, y, z \in B$. In this case ω is called a symplectic form, and the pair (B, ω) is called a symplectic subalgebra.

Lemma 5. Let (B, ω) be a symplectic subalgebra of \mathbb{M} . Then B is isomorphic to one of the following subalgebras:

1. The subalgebra with the base x, y' .
2. The subalgebra with the base h, x .

In these cases every non-degenerate skew-symmetric bilinear form is symplectic.

3. The subalgebra $M(4)$ with the base h, x, y', z . In this case non-degenerate skew-symmetric bilinear form is symplectic if and only if it satisfies

$$\omega(y', h) = 2\omega(x, z).$$

PROOF. Let a pair (B, ω) be a symplectic subalgebra of \mathbb{M} . Since ω is non-degenerate, the dimension of B is even. In [23], it was proved that the dimension of the maximal subalgebra in \mathbb{M} is equal to 5. Thus, we have two options for the dimension of B : 2 or 4. Let $\dim B = 2$. In this case, B is either abelian or non-abelian solvable Lie algebra. In the first case, B is isomorphic to the subalgebra with the base x, y' , in the second case—to the subalgebra with the base h, x . Clearly, in the both cases every non-degenerate skew-symmetric bilinear form is symplectic.

Let $\dim B = 4$. By Lemma 3, B is isomorphic to $M(4)$. Let ω be a symplectic form on B . From the condition

$$\omega(xz, h) + \omega(zh, x) + \omega(hx, z) = 0$$

we get that $\omega(y', h) = 2\omega(x, z)$. It is easy to see that the last equality is enough for a non-degenerate skew-symmetric bilinear form ω to be symplectic. \square

§5. The semisimple case.

Consider that $D(M\mathbb{M})$ is a semisimple Malcev algebra. Then by Lemma 1 $D(\mathbb{M}) = M_1 \oplus M_2$ is a direct sum of ideals where each of M_k ($k = 1, 2$) is isomorphic to \mathbb{M} . It is clear that $Q(M_1, M_2) = 0$ and thus the restriction Q_k of the form Q on M_k is a non-degenerate associative form.

Since $\mathbb{M}^* \subseteq \mathbb{M} + M_1$, then for all $f \in \mathbb{M}^*$ we have $f + a \in M_1$, where $a \in \mathbb{M}$. Let us define a mapping $\phi_1 : \mathbb{M}^* \rightarrow \mathbb{M}$ by $\phi_1(f) = -a$. The mapping ϕ_1 is a well-defined homomorphism of algebras. Since M^* is a dual space for M , there is an element $r_1 = \sum_i a_i \otimes b_i \in \mathbb{M} \otimes \mathbb{M}$, such that $\phi_1(f) = \sum_i f(a_i)b_i$.

Similarly, there is an element $r_2 = \sum_i c_i \otimes d_i \in \mathbb{M} \otimes \mathbb{M}$ such that the mapping $\phi_2 : \mathbb{M}^* \rightarrow \mathbb{M}$ defined by $\phi_2(f) = \sum_i f(c_i)d_i$ is a homomorphism of algebras with $f - \phi_2(f) \in M_2$. Since ϕ_1 and ϕ_2 are homomorphisms, then r_1 and r_2 are solutions of (5).

For all $f, g \in \mathbb{M}^*$ we have $Q(f - \phi_1(f), g - \phi_2(g)) = 0$. Hence,

$$\sum_i \langle f \otimes g, a_i \otimes b_i + d_i \otimes c_i \rangle = 0.$$

Consequently,

$$r_1 + \tau(r_2) = 0. \tag{10}$$

Also we have $(f - \phi_1(f))(g - \phi_2(g)) = 0$. Therefore,

$$fg - f \lhd \phi_2(g) - \phi_1(f) \rhd g = 0.$$

The last equality means that for all $a \in \mathbb{M}$

$$fg(a) = \sum_i f(g(c_i)d_i a) + g(f(a_i)ab_i).$$

$$\langle f \otimes g, \Delta(a) \rangle = \langle f \otimes g, \sum_i d_i a \otimes c_i + a_i \otimes ab_i \rangle.$$

Considering (10) we finally obtain

$$\langle f \otimes g, \Delta(a) \rangle = -\langle f \otimes g, \sum_i a_i a \otimes b_i - a_i \otimes ab_i \rangle.$$

In other words, we have proved that $\Delta(a) = -(\sum_i a_i a \otimes b_i - a_i \otimes ab_i) = -\Delta_{r_1}(a)$. Similarly, one can prove $\Delta(a) = -\Delta_{r_2}(a)$.

Let $r = r_1$. Then $\Delta = -\Delta_r$. Since \mathbb{M}^* is an anticommutative algebra, then $Q(fg + gf, a) = 0$ for all $a \in \mathbb{M}$. Therefore,

$$0 = \langle f \otimes g, -[r, a] - \tau([r, a]) \rangle = -\langle f \otimes g, [r + \tau(r), a] \rangle.$$

Thus $[r + \tau(r), a] = 0$ for all $a \in \mathbb{M}$. Put $s = \frac{1}{2}(r + \tau(r))$, $n = \frac{1}{2}(r - \tau(r))$. Then $r = s + n$, n is a skew-symmetric element, and $\Delta_r = \Delta_n$.

Lemma 6. Let K be an arbitrary Malcev algebra, $r \in K \otimes K$. Define $s = \frac{1}{2}(r + \tau(r))$, $n = \frac{1}{2}(r - \tau(r))$. Suppose r is a solution to the classical Yang-Baxter equation $C_K(r) = 0$ with $[s, a] = 0$ for all $a \in K$. Then the pair (K, Δ_r) is a Malcev bialgebra.

Proof. Since $[s, a] = 0$ for all $a \in K$, then $\Delta_r = \Delta_n$. It is enough to prove the pair (K, Δ_n) to be a Malcev bialgebra. Since $\tau(n) = -n$, then by Theorem 2 it is enough to prove that $C_K(n)$ satisfies (9).

Let $s = \sum_i a_i \otimes b_i$, $n = \sum_i p_i \otimes q_i$. Substituting the sum $s + n$ for r in $C_K(r) = 0$, we obtain

$$\begin{aligned} \sum_{ij} (a_i a_j \otimes b_i \otimes b_j - a_i \otimes a_j b_i \otimes b_j + a_i \otimes a_j \otimes b_i b_j) + (a_i p_j \otimes b_i \otimes q_j - a_i \otimes p_j b_i \otimes q_j + a_i \otimes p_j \otimes b_i q_j) + \\ + (p_i a_j \otimes q_i \otimes b_j - p_i \otimes a_j q_i \otimes b_j + p_i \otimes a_j \otimes q_i b_j) + C_K(n) = 0. \end{aligned}$$

Consider $\sum_i a_i p_j \otimes b_i \otimes q_j$. Bearing in mind that $\sum_i a_i a \otimes b_i + a_i \otimes b_i b = 0$, we obtain

$$\sum_i a_i p_j \otimes b_i \otimes q_j = - \sum_i a_i \otimes b_i p_j \otimes q_j = \sum_{ij} a_i \otimes p_j b_i \otimes q_j$$

for all $b \in K$. Similarly,

$$\sum_{ij} p_i \otimes a_j \otimes q_i b_j = \sum_{ij} p_i \otimes a_j q_i \otimes b_j.$$

and

$$\sum_{ij} a_i a_j \otimes b_i \otimes b_j = \sum_{ij} a_i \otimes a_j b_i \otimes b_j.$$

Therefore,

$$\sum_{ij} a_i \otimes a_j \otimes b_i b_j + a_i \otimes p_j \otimes b_i q_j + p_i a_j \otimes q_i \otimes b_j + C_K(n) = 0.$$

Consider $\sum_{ij} a_i \otimes p_j \otimes b_i q_j$. Taking into account that $\tau(n) = -n$ and $\sum_i a_i a \otimes b_i + a_i \otimes b_i b = 0$ for all $b \in K$ we have

$$\begin{aligned} \sum_{ij} a_i \otimes p_j \otimes b_i q_j &= - \sum_{ij} a_i q_j \otimes p_j \otimes b_i = \sum_{ij} a_i p_j \otimes q_j \otimes b_i = \\ &= - \sum_{ij} p_j a_i \otimes q_j \otimes b_i = - \sum_{ij} p_i a_j \otimes q_i \otimes b_j. \end{aligned}$$

Hence we finally obtain

$$\sum_{ij} a_i \otimes a_j \otimes b_i b_j + C_K(n) = 0. \quad (11)$$

It follows from (11) that $C_K(n) = - \sum_i a_i \otimes a_j \otimes b_i b_j$. Plug the expression obtained into (9). For all $a, b \in K$ we have

$$\begin{aligned} (C_K(n)(1 \otimes b \otimes 1))(1 \otimes a \otimes 1) &= - \sum_{ij} a_i \otimes (a_j b) a \otimes b_i b_j = - \sum_{ij} a_i \otimes a_j \otimes b_i ((b_j a) b) = \\ &= \sum_{ij} a_i \otimes a_j \otimes ((b_j a) b) b_i, \\ C_K(n)(ab \otimes 1 \otimes 1) &= - \sum_{ij} a_i (ab) \otimes a_j \otimes b_i b_j = \sum_{ij} a_i \otimes a_j \otimes (b_i (ab)) b_j = \end{aligned}$$

$$\begin{aligned}
&= - \sum_{ij} a_i \otimes a_j \otimes ((ab)b_i)b_j, \\
(C_K(n)(1 \otimes 1 \otimes a))(1 \otimes 1 \otimes b) &= - \sum_{ij} a_i \otimes a_j \otimes ((b_i b_j)a)b, \\
C_K(n)(b \otimes 1 \otimes a) &= - \sum_{ij} a_i b \otimes a_j \otimes (b_i b_j)a = \sum_{ij} a_i \otimes a_j \otimes ((b_i b_j)b_j)a = \\
&\quad - \sum_{ij} a_i \otimes a_j \otimes ((bb_i)b_j)a, \\
C_K(n)(a \otimes b \otimes 1) &= - \sum_{ij} a_i a \otimes a_j b \otimes b_i b_j = - \sum_{ij} a_i \otimes a_j \otimes (b_i a)(b_j b) = \\
&= - \sum_{ij} a_i \otimes a_j \otimes (ab_i)(bb_j),
\end{aligned}$$

Finally, by (8), we obtain

$$\begin{aligned}
&(C_K(r)(1 \otimes b \otimes 1))(1 \otimes a \otimes 1) - C_K(r)(ab \otimes 1 \otimes 1) - (C_K(r)(1 \otimes 1 \otimes a))(1 \otimes 1 \otimes b) - \\
&\quad - C_K(r)(b \otimes 1 \otimes a) + C_K(r)(a \otimes b \otimes 1) = \\
&= \sum_{ij} a_i \otimes a_j \otimes (((b_j a)b)b_i + ((ab)b_i)b_j + ((b_i b_j)a)b + ((bb_i)b_j)a - (ab_i)(bb_j)) = 0.
\end{aligned}$$

□

Lemma 7. If

$$l = h \otimes s_1 + x \otimes s_2 + x' \otimes s_3 + y \otimes s_4 + y' \otimes s_5 + z \otimes s_6 + z' \otimes s_7 \in \mathbb{M} \otimes \mathbb{M}$$

is such an element that $[l, a] = 0$ for all $a \in \mathbb{M}$ then

$$l = c \left(\frac{1}{2} h \otimes h + x \otimes x' + x' \otimes x + y \otimes y' + y' \otimes y + z \otimes z' + z' \otimes z \right)$$

for some $c \in F$.

Proof. From the condition $[l, h] = 0$ we obtain $s_1 h = 0$ and $-2x \otimes s_2 + x \otimes s_2 h = 0$. Therefore, $s_1 = ch$ for some $c \in F$ and $s_2 = \beta_1 x' + \beta_2 y' + \beta_3 z'$.

It follows from the condition $[l, x] = 0$ that $2cx \otimes h + x \otimes s_2 x = 0$. Hence, bearing in mind the condition obtained for s_2 , we finally get $s_2 = 2cx'$. Similarly one can prove that $s_3 = 2cx$, $s_4 = 2cy'$, $s_5 = 2cy$, $s_6 = 2cz'$, $s_7 = 2cz$. □

Since $[s, a] = 0$ for all $a \in \mathbb{M}$, then by Lemma 7

$$s = c \left(\frac{1}{2} h \otimes h + x \otimes x' + x' \otimes x + y \otimes y' + y' \otimes y + z \otimes z' + z' \otimes z \right).$$

If $c = 0$ then $\tau(r) = -r$, and by (10) $r_1 = r_2$ and $\phi_1 = \phi_2$. But in this case $f + \phi(f) \in M_1 \cap M_2 = 0$ for all $f \in \mathbb{M}^*$.

Hence $c \neq 0$, so we may assume that $c = \frac{1}{2}$. Then for r we have

$$\begin{aligned}
r &= h \otimes \left(\frac{1}{4} h + \alpha_{12} x + \alpha_{13} x' + \alpha_{14} y + \alpha_{15} y' + \alpha_{16} z + \alpha_{17} z' \right) + \\
&\quad + x \otimes (-\alpha_{12} h + \alpha_{23} x' + \alpha_{24} y + \alpha_{25} y' + \alpha_{26} z + \alpha_{27} z') +
\end{aligned}$$

$$\begin{aligned}
& +x' \otimes (-\alpha_{13}h' + (1 - \alpha_{23})x + \alpha_{34}y + \alpha_{35}y' + \alpha_{36}z + \alpha_{37}z') + \\
& +y \otimes (-\alpha_{14}h - \alpha_{24}x - \alpha_{34}x' + \alpha_{45}y' + \alpha_{46}z + \alpha_{47}z') + \\
& +y' \otimes (-\alpha_{15}h - \alpha_{25}x - \alpha_{35}x' + (1 - \alpha_{45})y + \alpha_{56}z + \alpha_{57}z') + \\
& +z \otimes (-\alpha_{16}h - \alpha_{26}x - \alpha_{36}x' - \alpha_{46}y - \alpha_{56}y' + \alpha_{27}z') + \\
& +z' \otimes (-\alpha_{17}h - \alpha_{27}x - \alpha_{37}x' - \alpha_{47}y - \alpha_{57}y' + (1 - \alpha_{27})z).
\end{aligned}$$

Let us consider Λ to be the following matrix

$$\Lambda = \begin{pmatrix} \frac{1}{4} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} & \alpha_{17} \\ -\alpha_{12} & 0 & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} & \alpha_{27} \\ -\alpha_{13} & 1 - \alpha_{23} & 0 & \alpha_{34} & \alpha_{35} & \alpha_{36} & \alpha_{37} \\ -\alpha_{14} & -\alpha_{24} & -\alpha_{34} & 0 & \alpha_{45} & \alpha_{46} & \alpha_{47} \\ -\alpha_{15} & -\alpha_{25} & -\alpha_{35} & 1 - \alpha_{45} & 0 & \alpha_{56} & \alpha_{57} \\ -\alpha_{16} & -\alpha_{26} & -\alpha_{36} & -\alpha_{46} & -\alpha_{56} & 0 & \alpha_{67} \\ -\alpha_{17} & -\alpha_{27} & -\alpha_{37} & -\alpha_{47} & -\alpha_{57} & 1 - \alpha_{67} & 0 \end{pmatrix}. \quad (12)$$

We will use the following notations: $a_1 = h$, $a_2 = x$, $a_3 = x'$, $a_4 = y$, $a_5 = y'$, $a_6 = z$, $a_7 = z'$. Then $a = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ is a basis of \mathbb{M} . Let γ_{ij}^k be the structure constants of \mathbb{M} with respect to a_1, \dots, a_7 . Put $\Gamma_k = (\gamma_{ij}^k)_{i,j=1,\dots,n}$. Then

$$\Gamma_1 = e_{23} - e_{32} + e_{45} - e_{54} + e_{67} - e_{76},$$

$$\Gamma_2 = 2e_{12} - 2e_{21} - 2e_{57} + 2e_{75}, \Gamma_3 = -2e_{13} + 2e_{31} + 2e_{46} - 2e_{64}, \Gamma_4 = 2e_{14} - 2e_{41} + 2e_{37} - 2e_{73},$$

$$\Gamma_5 = 2e_{15} - 2e_{51} + 2e_{26} - 2e_{62}, \Gamma_6 = 2e_{16} - 2e_{61} - 2e_{35} + 2e_{53}, \Gamma_7 = -2e_{17} + 2e_{71} - 2e_{24} + 2e_{42}.$$

We have proved that in order to describe all Malcev bialgebra structures on \mathbb{M} we should find all $r \in \mathbb{M} \otimes \mathbb{M}$ such that $r + \tau(r) \neq 0$, $[r + \tau(r), a] = 0$ for all $a \in \mathbb{M}$ and $C_{\mathbb{M}}(r) = 0$. To proceed in this direction, we need some properties of the mapping ϕ_1 .

Lemma 8. Algebras \mathbb{M}^* and \mathbb{M} are not isomorphic.

Proof. Assume the converse. Then ϕ_1 is an isomorphism of algebras and Λ is a non-degenerate matrix.

From (4) we obtain (using GAP)

$$\sum_l 2(\Lambda \Gamma_l)_{kl} + (\Lambda^\tau \Gamma_k)_{ll} = 0. \quad (13)$$

Putting $k = 1, 2, \dots, 7$ one by one into (13), we get

$$(2\alpha_{23} - 1) + (2\alpha_{45} - 1) + (2\alpha_{67} - 1) = 0. \quad (14)$$

$$\alpha_{12} - \alpha_{57} = 0. \quad (15)$$

$$\alpha_{13} - \alpha_{46} = 0. \quad (16)$$

$$\alpha_{14} + \alpha_{37} = 0. \quad (17)$$

$$\alpha_{15} + \alpha_{26} = 0. \quad (18)$$

$$\alpha_{16} - \alpha_{35} = 0. \quad (19)$$

$$\alpha_{17} - \alpha_{24} = 0. \quad (20)$$

We want to show that there exists an element $p \neq 0$ in \mathbb{M} such that $\Delta_r(p) = [r, p] = 0$. In this case, the space $\{f \in \mathbb{M}^* | f(p) = 0\}$ is a proper ideal of \mathbb{M}^* , which contradicts the simplicity of \mathbb{M}^* . We have

$$\begin{aligned} [r, h] = & -4\alpha_{12}(h \otimes x - x \otimes h) + 4\alpha_{13}(h \otimes x' - x' \otimes h) - 4\alpha_{14}(h \otimes y - y \otimes h) + 4\alpha_{15}(h \otimes y' - y' \otimes h) - \\ & -4\alpha_{16}(h \otimes z - z \otimes h) + 4\alpha_{17}(h \otimes z' - z' \otimes h) - 4\alpha_{24}(x \otimes y - y \otimes x) - 4\alpha_{26}(x \otimes z - z \otimes x) + \\ & + 4\alpha_{35}(x' \otimes y' - y' \otimes x') + 4\alpha_{37}(x' \otimes z' - z' \otimes x') - 4\alpha_{46}(y \otimes z - z \otimes x) + 4\alpha_{57}(y' \otimes z' - z' \otimes y'). \end{aligned}$$

$$\begin{aligned} [r, x] = & 4\alpha_{13}(x \otimes x' - x' \otimes x) - 4\alpha_{14}(h \otimes z' - z' \otimes h - x \otimes y + y \otimes x) + 4\alpha_{15}(x \otimes y' - y' \otimes x) + \\ & + 4\alpha_{16}(h \otimes y' - y' \otimes h + x \otimes z - z \otimes x) + 4\alpha_{17}(x \otimes z' - z' \otimes x) + (2\alpha_{23} - 1)(h \otimes x - x \otimes h) - \\ & - 4\alpha_{24}(x \otimes z' - z' \otimes x) + 4\alpha_{26}(x \otimes y' - y' \otimes x) - 2\alpha_{34}(h \otimes y - y \otimes h + 2x' \otimes z' - 2z' \otimes x') - \\ & - 2\alpha_{35}(h \otimes y' - y' \otimes h) + -2\alpha_{36}(h \otimes z - z \otimes h - 2x' \otimes y' + 2y' \otimes x') - 2\alpha_{37}(h \otimes z' - z' \otimes h) + \\ & + 2(2\alpha_{45} - 1)(y' \otimes z' - z' \otimes y') + 4\alpha_{46}(z \otimes z' - z' \otimes z + y \otimes y' - y' \otimes y) + 2(2\alpha_{67} - 1)(y' \otimes z' - z' \otimes y'). \end{aligned}$$

$$\begin{aligned} [r, x'] = & 4\alpha_{12}(x \otimes x' - x' \otimes x) - 4\alpha_{14}(x' \otimes y - y \otimes x') + 4\alpha_{15}(h \otimes z - z \otimes h - x' \otimes y' + y' \otimes x') - \\ & - 4\alpha_{16}(x' \otimes z - z \otimes x') - 4\alpha_{17}(h \otimes y - y \otimes h + x' \otimes z' - z' \otimes x') + (2\alpha_{23} - 1)(h \otimes x' - x' \otimes h) + \\ & + 4\alpha_{24}(h \otimes y - y \otimes h) + 2\alpha_{25}(h \otimes y' - y' \otimes h + 2x \otimes z - 2z \otimes x) + 4\alpha_{26}(h \otimes z - z \otimes h) + \\ & + 2\alpha_{27}(h \otimes z' - z' \otimes h + 2x \otimes y - 2y \otimes x) + 4\alpha_{35}(x' \otimes z - z \otimes x') - 4\alpha_{37}(x' \otimes y - y \otimes x') + \\ & + 2(2\alpha_{45} - 1)(y \otimes z - z \otimes y) + 4\alpha_{57}(z \otimes z' - z' \otimes z + y \otimes y' - y' \otimes y) + 2(2\alpha_{67} - 1)(y \otimes z - z \otimes y). \end{aligned}$$

$$\begin{aligned} [r, y] = & 4\alpha_{12}(h \otimes z' - z' \otimes h - x \otimes y + y \otimes x) - 4\alpha_{13}(x' \otimes y - y \otimes x') + 4\alpha_{15}(y \otimes y' - y' \otimes y) - \\ & - 4\alpha_{16}(h \otimes x' - x' \otimes h - y \otimes z + z \otimes y) + \alpha_{17}(y \otimes z' - z' \otimes y) - 2(2\alpha_{23} - 1)(x' \otimes z' - z' \otimes x') - \\ & - 4\alpha_{24}(y \otimes z' - z' \otimes y) + 2\alpha_{25}(h \otimes x - x \otimes h - 2y' \otimes z' + 2z' \otimes y') - \\ & - 4\alpha_{26}(z \otimes z' - z' \otimes z + x \otimes x' - x' \otimes x) + 2\alpha_{35}(h \otimes x' - x' \otimes h) + (2\alpha_{45} + 1)(h \otimes y - y \otimes h) + \\ & + 4\alpha_{46}(x' \otimes y - y \otimes x') - 2\alpha_{56}((h \otimes z - z \otimes h - 2x' \otimes y' + 2y' \otimes x') - 2\alpha_{57}(h \otimes z' - z' \otimes h) - \\ & - 2(2\alpha_{67} - 1)(x' \otimes z' - z' \otimes x'). \end{aligned}$$

$$\begin{aligned} [r, y'] = & 4\alpha_{12}(x \otimes y' - y' \otimes x) - 4\alpha_{13}(h \otimes z - z \otimes h - x' \otimes y' + y' \otimes x') + 4\alpha_{14}(y \otimes y' - y' \otimes y) - \\ & - 4\alpha_{16}(y' \otimes z - z \otimes y') + 4\alpha_{17}(h \otimes x - x \otimes h + y' \otimes z' - z' \otimes y') - 2(2\alpha_{23} - 1)(x \otimes z - z \otimes x) - \\ & - 2\alpha_{24}(h \otimes x - x \otimes h) - 4\alpha_{34}(h \otimes x' - x' \otimes h - y \otimes z + z \otimes y) + 4\alpha_{35}(y' \otimes z - z \otimes y') - \\ & - 4\alpha_{37}(x \otimes x' - x' \otimes x + z \otimes z' - z' \otimes z) + (2\alpha_{45} - 1)(h \otimes y' - y' \otimes h) + 2\alpha_{46}(h \otimes z - z \otimes h) + \\ & + 2\alpha_{47}(h \otimes z - z \otimes h - x \otimes y + y \otimes x) - 4\alpha_{57}(x \otimes y' - y' \otimes x) - (2\alpha_{67} - 1)(x \otimes z - z \otimes x). \end{aligned}$$

$$\begin{aligned}
[r, z] = & -4\alpha_{12}(h \otimes y' - y' \otimes x + x \otimes z - z \otimes x) - 4\alpha_{13}(x' \otimes z - z \otimes x') + \\
& + 4\alpha_{14}(h \otimes x' - x' \otimes h - y \otimes z + z \otimes y) - 4\alpha_{15}(y' \otimes z - z \otimes y') + 4\alpha_{17}(z \otimes z' - z' \otimes z) + \\
& + 2(2\alpha_{23} - 1)(x' \otimes y' - y' \otimes x') + 4\alpha_{24}(x \otimes x' - x' \otimes x + y \otimes y' - y' \otimes y) - 4\alpha_{26}(y' \otimes z - z \otimes y') + \\
& + 2\alpha_{27}(h \otimes x - x \otimes h - 2y' \otimes z' + 2z' \otimes y') + 2\alpha_{37}(h \otimes x' - x' \otimes h) + 2(2\alpha_{45} - 1)(x' \otimes y' - y' \otimes x') + \\
& + 4\alpha_{46}(x' \otimes z - z \otimes x') + 2\alpha_{47}(h \otimes y - y \otimes h + 2x' \otimes z' - 2z' \otimes x') + 2\alpha_{57}(h \otimes y' - y' \otimes h) + \\
& + (2\alpha_{67} - 1)(h \otimes z - z \otimes h).
\end{aligned}$$

$$\begin{aligned}
[r, z'] = & 4\alpha_{12}(x \otimes z' - z' \otimes x) + 4\alpha_{13}(h \otimes y - y \otimes h + x' \otimes z' - z' \otimes x') + 4\alpha_{14}(y \otimes z' - z' \otimes y) - \\
& - 4\alpha_{15}(h \otimes x - x \otimes h - y' \otimes z' + z' \otimes y') + 4\alpha_{16}(z \otimes z' - z' \otimes z) + 2(2\alpha_{23} - 1)(x \otimes y - y \otimes x) - \\
& - 2\alpha_{26}(h \otimes x - x \otimes h) + 4\alpha_{35}(x \otimes x' - x' \otimes x + y \otimes y' - y' \otimes y) - 2\alpha_{36}(h \otimes x' - x' \otimes h - 2y \otimes z + 2z \otimes y) \\
& + 4\alpha_{37}(y \otimes z' - z' \otimes y) + 2(2\alpha_{45} - 1)(x \otimes y - y \otimes x) - 2\alpha_{46}(h \otimes y - y \otimes h) - \\
& - 2\alpha_{56}(h \otimes y' - y' \otimes h + 2x \otimes z - 2z \otimes x) - 4\alpha_{57}(x \otimes z' - z' \otimes x) + (2\alpha_{67} - 1)(h \otimes z' - z' \otimes h).
\end{aligned}$$

Take $p = \alpha_{12}x - \alpha_{13}x' + \alpha_{14}y - \alpha_{15}y' + \alpha_{16}z - \alpha_{17}z'$, then

$$\begin{aligned}
[r, \alpha_{12}x - \alpha_{13}x' + \alpha_{14}y - \alpha_{15}y' + \alpha_{16}z - \alpha_{17}z'] = & \\
= & (\alpha_{12}(2\alpha_{23} - 1) + 2\alpha_{14}\alpha_{25} + 2\alpha_{16}\alpha_{27})(h \otimes x - x \otimes h - 2y' \otimes z' + 2z' \otimes y') + \\
& + (\alpha_{14}(2\alpha_{45} - 1) - 2\alpha_{12}\alpha_{34} + 2\alpha_{16}\alpha_{47})(h \otimes y - y \otimes h + 2x' \otimes z' - 2z' \otimes x') + \\
& + (\alpha_{16}(2\alpha_{67} - 1) - 2\alpha_{12}\alpha_{36} - 2\alpha_{14}\alpha_{56})(h \otimes z - z \otimes h - 2x' \otimes y' + 2y' \otimes x') + \\
& + (2\alpha_{15}\alpha_{34} + 2\alpha_{17}\alpha_{36} - \alpha_{13}(2\alpha_{23} - 1))(h \otimes x' - x' \otimes h - 2y \otimes z + 2z \otimes x) + \\
& + (2\alpha_{17}\alpha_{56} - 2\alpha_{13}\alpha_{25} - \alpha_{15}(2\alpha_{45} - 1))(h \otimes y' - y' \otimes h + 2x \otimes z - 2z \otimes x) + \\
& + (-2\alpha_{13}\alpha_{27} - 2\alpha_{15}\alpha_{47} - \alpha_{17}(2\alpha_{67} - 1))(h \otimes z' - z' \otimes h - 2x \otimes y + 2y \otimes x).
\end{aligned}$$

In (3) put $k = 7, s = 1, n = 5$ to get

$$\begin{aligned}
0 = & \alpha_{14}\alpha_{25} + \alpha_{16}\alpha_{27} - \alpha_{12}\alpha_{45} - \alpha_{12}\alpha_{67} + \alpha_{12} = \alpha_{14}\alpha_{25} + \alpha_{16}\alpha_{27} + \alpha_{12}((\alpha_{45} - \frac{1}{2}) + (\alpha_{67} - \frac{1}{2})) = \\
= & \alpha_{14}\alpha_{25} + \alpha_{16}\alpha_{27} + \alpha_{12}(\alpha_{23} - \frac{1}{2}).
\end{aligned}$$

Thus, the coefficient of the first summand of $[r, p]$ is equal to zero. Similarly, putting $\{k = 7, s = 1, n = 3\}$, $\{k = 3, s = 1, n = 5\}$, $\{k = 6, s = 1, n = 4\}$, $\{k = 6, s = 1, n = 2\}$, $\{k = 2, s = 1, n = 4\}$, we obtain $[r, p] = 0$. \square

By Lemma 2 the space

$$V = \{a \in \mathbb{M} \mid a + f \in M_2 \text{ for some } f \in \mathbb{M}^*\}$$

is a subbialgebra of (\mathbb{M}, Δ) and $M_2V^\perp = 0$. Then $V^\perp \subseteq M_1$. By Lemma 8 $V \neq \mathbb{M}$. In [23] it was proved that any maximal subalgebra in \mathbb{M} is isomorphic to the algebra $M(5)$ with base h, x, x', y', z .

On the other hand, if $\dim V \leq 3$ then $\dim(\mathbb{M}^* \cap M_1) \geq 4$. But $Q(\mathbb{M}^*, \mathbb{M}^*) = 0$ that is impossible since Q_1 is non-degenerate. So, we have two options: either $\dim V = 4$ or $\dim V = 5$.

Lemma 9. The dimension of V does not equal 5.

PROOF. Suppose that $\dim V = 5$. We can assume, that the elements h, x, x', y', z form the basis of V . By Lemma 2 the space

$$V^\perp = \{f \in M^* \mid f(a) = 0 \text{ for every } a \in V\}$$

satisfies $M_2 V^\perp = 0$. So, $V^\perp \subseteq M_1$ and, therefore, $V^\perp = \ker \phi_1$. If $h^*, x^*, x'^*, y^*, y'^*, z^*, z'^*$ is a dual base of \mathbb{M}^* to the base h, x, x', y, y', z, z' then the functionals y^* and z'^* form a base of V^\perp . In this case, some entries of the matrix (12) are zero:

$$\alpha_{14} = \alpha_{24} = \alpha_{34} = \alpha_{45} = \alpha_{46} = \alpha_{47} = 0, \alpha_{j7} = 0$$

for all $i, j = 1, \dots, 6$. Thus

$$\Lambda = \begin{pmatrix} \frac{1}{4} & \alpha_{12} & \alpha_{13} & 0 & \alpha_{15} & \alpha_{16} & 0 \\ -\alpha_{12} & 0 & \alpha_{23} & 0 & \alpha_{25} & \alpha_{26} & 0 \\ -\alpha_{13} & 1 - \alpha_{23} & 0 & 0 & \alpha_{35} & \alpha_{36} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\alpha_{15} & -\alpha_{25} & -\alpha_{35} & 1 & 0 & \alpha_{56} & 0 \\ -\alpha_{16} & -\alpha_{26} & -\alpha_{36} & 0 & -\alpha_{56} & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (21)$$

Since $\dim V = 5$, then the rank of Λ is equal to 5.

Now, if we put Λ into (3), we obtain the following equalities (by means of GAP):

$$4\alpha_{12}\alpha_{13} - \alpha_{23}(\alpha_{23} - 1) = 0 \quad \{n = 3, k = 2, s = 1\}; \quad (22)$$

$$4\alpha_{12}\alpha_{15} + 2\alpha_{12}\alpha_{26} - \alpha_{25}(\alpha_{23} - 1) = 0 \quad \{n = 5, k = 2, s = 1\}; \quad (23)$$

$$2\alpha_{12}\alpha_{36} + 2\alpha_{16}(\alpha_{23} - 1) - \alpha_{35}(\alpha_{23} - 1) = 0 \quad \{n = 5, k = 3, s = 1\}; \quad (24)$$

$$-4\alpha_{12}\alpha_{35} + 2\alpha_{26}(\alpha_{23} - 1) = 0 \quad \{n = 5, k = 3, s = 2\}; \quad (25)$$

$$2\alpha_{13}\alpha_{25} - 2\alpha_{15}\alpha_{23} - \alpha_{23}\alpha_{26} = 0 \quad \{n = 6, k = 2, s = 1\}; \quad (26)$$

$$-4\alpha_{13}\alpha_{16} + 2\alpha_{13}\alpha_{35} - \alpha_{23}\alpha_{36} = 0 \quad \{n = 6, k = 3, s = 1\}; \quad (27)$$

$$-4\alpha_{13}\alpha_{26} + 2\alpha_{23}\alpha_{35} = 0 \quad \{n = 6, k = 3, s = 2\}; \quad (28)$$

$$-4\alpha_{15}\alpha_{16} + 2\alpha_{15}\alpha_{35} - 2\alpha_{16}\alpha_{26} - \alpha_{25}\alpha_{36} + \alpha_{26}\alpha_{35} = 0 \quad \{n = 6, k = 5, s = 1\}; \quad (29)$$

$$-4\alpha_{15}\alpha_{26} + 2\alpha_{25}\alpha_{35} - 2\alpha_{26}^2 = 0 \quad \{n = 6, k = 5, s = 2\}; \quad (30)$$

$$-4\alpha_{16}\alpha_{35} - 2\alpha_{26}\alpha_{36} + 2\alpha_{35}^2 = 0 \quad \{n=6, k=5, s=3\}; \quad (31)$$

Other relations coming from another values of n, k, s follow from the given equalities. Consider a matrix

$$\Lambda_1 = \begin{pmatrix} \frac{1}{4} & \alpha_{12} & \alpha_{13} & 0 & \alpha_{15} & \alpha_{16} & 0 \\ -\alpha_{12} & 0 & \alpha_{23} & 0 & \alpha_{25} & \alpha_{26} & 0 \\ -\alpha_{13} & 1 - \alpha_{23} & 0 & 0 & \alpha_{35} & \alpha_{36} & 0 \end{pmatrix}.$$

Let us prove that the rank of Λ_1 does not exceed 2. For this, consider a matrix

$$\Lambda_2 = \begin{pmatrix} \frac{1}{4} & \alpha_{12} & \alpha_{13} \\ -\alpha_{12} & 0 & \alpha_{23} \\ -\alpha_{13} & 1 - \alpha_{23} & 0 \end{pmatrix}.$$

Let

$$\Lambda_2^* = \begin{pmatrix} -\alpha_{23}(1 - \alpha_{13}) & \alpha_{13}(1 - \alpha_{23}) & \alpha_{12}\alpha_{23} \\ -\alpha_{13}\alpha_{23} & \alpha_{13}^2 & -(\frac{1}{4}\alpha_{23} + \alpha_{12}\alpha_{13}) \\ -\alpha_{12}(1 - \alpha_{23}) & -(\frac{1}{4}(1 - \alpha_{23}) + \alpha_{12}\alpha_{13}) & \alpha_{12}^2 \end{pmatrix}.$$

Then $\Lambda_2\Lambda_2^* = \det(\Lambda_2)E$.

Let V_2 and V_3 be second and third rows of Λ_2^* , respectively. It is easy to see that V_2 and V_3 can not be equal to zero simultaneously. By (22) $\det(\Lambda_2) = 0$, so $\Lambda_2\Lambda_2^* = 0$. In particular, $V_i\Lambda_2 = 0$, for $i = 2, 3$.

Let U_5 and U_6 be 5th and 6th columns of Λ_1 , respectively. Let us prove that $V_iU_j = 0$ for all $i = 2, 3, j = 5, 6$. We have

$$\begin{aligned} V_2U_5 &= -\alpha_{13}\alpha_{23}\alpha_{15} + \alpha_{13}^2\alpha_{25} - (\frac{1}{4}\alpha_{23} + \alpha_{12}\alpha_{13})\alpha_{35} \stackrel{(26)}{=} -\alpha_{13}\alpha_{23}\alpha_{15} + \alpha_{13}\alpha_{23}\alpha_{15} + \\ &+ \frac{1}{2}\alpha_{13}\alpha_{23}\alpha_{26} - (\frac{1}{4}\alpha_{23} + \alpha_{12}\alpha_{13})\alpha_{35} \stackrel{(28)}{=} \alpha_{35}(\frac{1}{4}\alpha_{23}(\alpha_{23} - 1) - \alpha_{12}\alpha_{13}) \stackrel{(22)}{=} 0. \end{aligned}$$

$$\begin{aligned} V_2U_6 &= -\alpha_{13}\alpha_{23}\alpha_{16} + \alpha_{13}^2\alpha_{26} - (\frac{1}{4}\alpha_{23} + \alpha_{12}\alpha_{13})\alpha_{36} \stackrel{(28)}{=} \alpha_{13}(-\alpha_{23}\alpha_{16} + \frac{1}{2}\alpha_{23}\alpha_{35} - \alpha_{12}\alpha_{36}) - \\ &- \frac{1}{4}\alpha_{23}\alpha_{36} \stackrel{(24)}{=} -\alpha_{13}\alpha_{16} + \frac{1}{2}\alpha_{13}\alpha_{35} - \frac{1}{4}\alpha_{23}\alpha_{36} \stackrel{(27)}{=} 0 \end{aligned}$$

Similarly, $V_3U_5 = V_3U_6 = 0$. It follows that the rows of Λ_1 are linearly dependent. Therefore the rank of Λ does not exceed 4. \square

Now we are ready to prove the main theorem in this section.

Theorem 5. Let \mathbb{M} be a simple non-Lie Malcev algebra over an algebraically closed field of characteristic not equal 2, 3. Then in a standard basis h, x, x', y, y', z, z' of the algebra \mathbb{M} the element

$$\begin{aligned} r_0 &= \alpha_{12}(h \otimes x - x \otimes h) + \alpha_{15}(h \otimes y' - y' \otimes h) + \alpha_{16}(h \otimes z - z \otimes h) + \\ &+ \alpha_{25}(x \otimes y' - y' \otimes x) - 2\alpha_{15}(x \otimes z - z \otimes x) + \alpha_{56}(y' \otimes z - z \otimes y') \end{aligned}$$

is a solution of the classical Yang-Baxter equation on \mathbb{M} . Moreover, the element

$$r = r_0 + \frac{1}{4}h \otimes h + x \otimes x' + y' \otimes y + z \otimes z'. \quad (32)$$

induces on \mathbb{M} a structure of a Malcev bialgebra with a semisimple Drinfeld double.

Conversely, let (\mathbb{M}, Δ) be a Malcev bialgebra with a semisimple Drinfeld double. Then $\Delta = -\Delta_r$, and one can choose a standard basis h, x, x', y, y', z, z' of \mathbb{M} in such a way that r has the form (32).

Proof. By Lemma 5 and (11) we obtain r_0 to be a solution to the classical Yang-Baxter equation on \mathbb{M} .

Let $r = r_0 + \frac{1}{4}h \otimes h + x \otimes x' + y' \otimes y + z \otimes z'$. For r to be a solution to the equation $C_{\mathbb{M}}(r) = 0$ it is necessary and sufficient that the equalities (22)–(31) hold. Since in our case $\alpha_{13} = \alpha_{35} = \alpha_{36} = 0$ and $\alpha_{23} = 1$, straightforward calculations (using GAP) show that (22)–(31) follow from $2\alpha_{15} + \alpha_{26} = 0$. Thus, by Lemma 6, the pair (\mathbb{M}, Δ_r) is a Malcev bialgebra. Since $r \neq -\tau(r)$, then the Drinfeld double $D(M)$ has to be a semisimple algebra.

Conversely, let (\mathbb{M}, Δ) be a Malcev bialgebra with a semisimple Drinfeld double. By Lemmas 8 and 9 the dimension of V equals 4. Thence, $\ker \phi_1 = V^\perp$. Then, by Lemma 3 one can choose a standard basis h, x, x', y, y', z, z' of \mathbb{M} in such a way that V has the base h, x, y', z . If $h^*, x^*, x'^*, y^*, y'^*, z^*, z'^*$ is the dual base to the base h, x, x', y, y', z, z' then the elements x'^*, y^*, z'^* form a base of V^\perp . Therefore,

$$\Lambda = \begin{pmatrix} \frac{1}{4} & \alpha_{12} & 0 & 0 & \alpha_{15} & \alpha_{16} & 0 \\ -\alpha_{12} & 0 & 1 & 0 & \alpha_{25} & \alpha_{26} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\alpha_{15} & -\alpha_{25} & 0 & 1 & 0 & \alpha_{56} & 0 \\ -\alpha_{16} & -\alpha_{26} & 0 & 0 & -\alpha_{56} & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From the equality (3) with $n = 6$, $k = 2$, $s = 1$ we obtain $2\alpha_{15} + \alpha_{26} = 0$. Hence r has the form (32). \square

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